Numerische
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Entropy stable shock capturing space–time discontinuous Galerkin schemes for systems of conservation laws

Andreas Hiltebrand · Siddhartha Mishra

Abstract We present a streamline diffusion shock capturing spacetime discontinuous Galerkin (DG) method to approximate nonlinear systems of conservation laws in several space dimensions. The degrees of freedom are in terms of the entropy variables and the numerical flux functions are the entropy stable finite volume fluxes. We show entropy stability of the (formally) arbitrarily high order accurate method for a general system of conservation laws. Furthermore, we prove that the approximate solutions converge to the entropy measure valued solutions for nonlinear systems of conservation laws. Convergence to entropy solutions for scalar conservation laws and for linear symmetrizable systems is also shown. Numerical experiments are presented to illustrate the robustness of the proposed schemes.

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1 Introduction

Many interesting problems in Physics and Engineering are modeled in terms of nonlinear partial differential equations termed as systems of conservation laws:

A. Hiltebrand
Seminar for Applied Mathematics (SAM), Department of Mathematics, ETH Zürich,
HG J 49, Zürich 8092, Switzerland
e-mail: andreas.hiltebrand@sam.math.ethz.ch

S. Mishra
Center of Mathematics for Applications (CMA), University of Oslo, P.O. Box 1053,
Blindern, Oslo 0316, Norway

S. Mishra (✉)
Seminar for Applied Mathematics (SAM), Department of Mathematics, ETH Zürich,
HG G 57.2, Zürich 8092, Switzerland
e-mail: smishra@sam.math.ethz.ch

$$\mathbf{U}_t + \sum_{k=1}^{d} \mathbf{F}^k(\mathbf{U})_{x_k} = 0, \quad (x, t) \in \Omega \times \mathbb{R}_+, \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^d (d = 1, 2, 3)$ is a bounded spatial domain and $\mathbf{U} : \Omega \mapsto \mathbb{R}^m$ is the vector of unknowns. $\mathbf{F}^k$ is the (smooth) flux vector in the k-th direction. The conservation law (1.1) is equipped with suitable initial and boundary conditions. Examples for systems of conservation laws include the Euler equations of gas dynamics, the shallow water equations of oceanography, the magnetohydrodynamics (MHD) equations of plasma physics and equations of nonlinear elasticity [6].

1.1 The continuous problem

The system of conservation laws (1.1) is termed *hyperbolic* if the directional Jacobians $\sum_{k=1}^{d} \partial_{\mathbf{U}} \mathbf{F}^k v_k$, have real eigenvalues for every normal direction v. It is well known that solutions of hyperbolic conservation laws contain discontinuities known as shock waves, even when the initial data is smooth. Thus, solutions of (1.1) are sought in the weak sense i.e, for every compactly supported test function $\boldsymbol{\varphi} \in (C_c^\infty(\Omega \times \mathbb{R}_+))^m$, $\mathbf{U} \in (L^1(\Omega \times \mathbb{R}_+))^m$ is said to be a weak solution of (1.1) if the following integral identity is satisfied:

$$\int_{\mathbb{R}_+} \int_{\Omega} \left(\langle \mathbf{U}, \boldsymbol{\varphi}_t \rangle + \sum_{d=1}^{k} \langle \mathbf{F}^k(\mathbf{U}), \boldsymbol{\varphi}_{x_k} \rangle \right) dx dt + \int_{\Omega} \langle \mathbf{U}(x, 0), \boldsymbol{\varphi}(x, 0) \rangle dx = 0. \tag{1.2}$$

Here, we have implicitly ignored boundary conditions by considering test functions that are compactly supported in the spatial domain Ω. Weak solutions are not necessarily unique. Additional admissibility criteria or *entropy conditions* need to be imposed in order to single out the physically relevant solutions of (1.1). We assume that there exists a strictly convex entropy function S and entropy flux functions Q^k such that the following compatibility conditions are satisfied,

$$\partial_{\mathbf{U}} Q^k = \langle \mathbf{V}, \partial_{\mathbf{U}} \mathbf{F}^k \rangle, \quad \forall k = 1, 2, \cdots, d. \tag{1.3}$$

Here, $\mathbf{V} = \partial_{\mathbf{U}} S$ is termed as the vector of entropy variables. The weak solution $\mathbf{U}$ of (1.1) is said to be the entropy solution [6] if it satisfies the entropy inequality,

$$S_t + \sum_{k=1}^{d} Q_{x_k}^k \leq 0, \tag{1.4}$$

in the sense of distributions.

Integrating the entropy inequality (1.4) in space results in the estimate,

$$\frac{d}{dt} \int_{\Omega} S(\mathbf{U}(x, t)) dx \leq 0. \tag{1.5}$$

This bound on the total entropy, together with the strict convexity of the entropy function, yields an L^2 stability estimate for the entropy solution **U** [6]. This "energy" estimate is currently the only available generic global a priori estimate for systems of conservation laws [6]. Furthermore, the entropy inequality incorporates appropriate small scale information such as vanishing diffusion, resistivity, etc [6].

1.2 Numerical schemes

Numerical schemes serve as one of the key tools in the study of systems of conservation laws. Finite volume schemes [23] that update the cell averages of the solution in terms of the numerical fluxes (obtained by the exact or approximate solutions of Riemann problems at cell interfaces) are one of the most popular design frameworks for robust numerical schemes. Higher-order spatial accuracy is obtained from a non-oscillatory piecewise polynomial reconstruction in each cell. Reconstruction procedures such as the second-order TVD [23], ENO [14] and WENO [25] are typically employed. Higher order temporal accuracy results from strong stability preserving (SSP) Runge–Kutta (RK) time integrators. An alternative to high-order finite volume methods is the discontinuous Galerkin finite element method [4,5]. At lowest (first) order, these methods reduce to the standard finite volume method. However, high-order accuracy is obtained by using piecewise polynomial test functions in each element. Limiters are employed to damp oscillations near shocks. Temporal accuracy is again increased by using SSP RK methods. High-order finite volume methods and RKDG methods have been very successful in carrying out realistic large scale simulations of conservation laws [26].

However, the rigorous analysis of numerical methods for systems of conservation laws is far from being complete. It is essential to remark that most of the rigorous numerical analysis results have been obtained for scalar conservation laws [13]. For these equations, the continuous problem is well-posed as the solution is *TVD* and satisfies an infinite number of entropy inequalities. First-order monotone finite volume schemes and second-order TVD limiter based schemes are shown to converge to entropy solution [13]. However, no rigorous global stability results are available for the ENO and WENO schemes, even for one-dimensional scalar conservation laws. Second-order TVD limiter based and arbitrary order TVB limiter based RKDG schemes are also shown to converge [5] for scalar problems. It is also possible to obtain rigorous convergence results for linear symmetrizable systems [11].

As global estimates on the total variation are not available for solutions of systems of conservation laws, the entropy estimate (1.5) and the resulting L^2 bound serve as the only available a priori estimates on the entropy solution. Therefore, we seek *entropy stable schemes* i.e, numerical schemes approximating (1.1) that satisfy a discrete version of the entropy inequality (1.4) and the resulting entropy estimate (1.5).

1.3 Entropy stable numerical schemes

The first entropy stable schemes for systems of conservation laws were introduced by Tadmor in [27,28]. These first order finite volume (finite difference) schemes are

based on judicious combination of *entropy conservative fluxes* and *numerical diffusion operators*. Arbitrary order entropy stable schemes (TeCNO schemes) for systems of conservation laws, based on high order entropy conservative fluxes [22] and ENO based numerical diffusion operators has been proposed very recently in [10,11]. However, these schemes, being finite difference schemes, are restricted to structured grids in several space dimensions. Furthermore, only the semi-discrete version of TeCNO schemes has been shown to be entropy stable so far.

In contrast to finite volume schemes, the theme of entropy stable finite element methods has been investigated more widely. In [17], the authors proposed a streamline diffusion based entropy stable spacetime finite element method. Similar spacetime streamline diffusion finite element methods for scalar conservation laws were proposed in [20,21] and a combined streamline diffusion and discontinuous Galerkin (DG) method for scalar conservation laws was proposed in [19]. More recently, entropy stable streamline diffusion and DG methods for systems of conservation laws were proposed by Barth in [1]. The major advantage of spacetime finite element methods is the fact that they are readily available for unstructured grids, unlike high-order ENO based finite volume methods. Furthermore, the spacetime formulation leads to entropy stability for fully discrete schemes.

1.4 Convergence to measure valued solutions

Entropy stability is an essential requirement for the design of efficient numerical schemes. However, it is not enough to conclude any sort of convergence result for the approximate solutions, generated by the entropy stable numerical scheme. Given the extreme difficulties of trying to prove wellposedness of weak solutions for nonlinear systems in several space dimensions and the corresponding difficulties in showing convergence of weak solutions for these equations, it might be necessary to weaken the notion of solutions even further.

To this end, we consider *entropy measure valued solutions* introduced by DiPerna in [7]. This concept of solutions is defined as,

Definition 1.1 (*Measure valued solutions*) Assuming that the conservation law (1.1) is equipped with an entropy formulation, a probability (non-negative with unit mass) measure $\boldsymbol{\mu}$, realized as a map:

$$\boldsymbol{\mu} : (x, t) \in \Omega \times \mathbb{R}_+ \mapsto Prob(\mathbb{R}^m),$$

for each x, t is a defined as a measure valued solution of the system (1.1) if it satisfies,

$$\int\limits_{\Omega} \int\limits_{\mathbb{R}_+} \left(\langle \langle \mathbf{U}, \boldsymbol{\mu}_{x,t} \rangle, \boldsymbol{\varphi}_t \rangle + \sum_{k=1}^{d} \langle \langle \mathbf{F}^k, \boldsymbol{\mu}_{x,t} \rangle, \boldsymbol{\varphi}_{x_k} \rangle \right) dx dt = 0, \qquad (1.6)$$

for all test functions $\boldsymbol{\varphi} \in (C_c^\infty(\Omega \times (0, \infty)))^m$. Here,

$$\langle \mathbf{g}, \boldsymbol{\mu}_{x,t} \rangle = \int\limits_{\mathbb{R}^m} \mathbf{g}(\lambda) d\boldsymbol{\mu}_{x,t}(\lambda).$$

Note that we realize the Young measure in terms of the entropy variables **V**. The corresponding Young measure for the conservative variables **U** can be realized as **U**($\boldsymbol{\mu}$), using the one-one mapping **U**(**V**).

Furthermore, as the system (1.1) is equipped with an entropy function S and entropy flux functions Q^k for $k = 1, 2, \cdots, d$, then $\boldsymbol{\mu}$ is defined to be an entropy measure valued solution of (1.1) if it is a measure valued solution as well as it satisfies,

$$\int_{\Omega} \int_{\mathbb{R}_+} \left(\langle S, \boldsymbol{\mu}_{x,t} \rangle \varphi_t + \sum_{k=1}^{d} \langle Q^k, \boldsymbol{\mu}_{x,t} \rangle \varphi_{x_k} \right) dx dt \geq 0, \tag{1.7}$$

for all non-negative test functions $0 \leq \varphi \in C_c^\infty(\Omega \times (0, \infty))$

It is well known that measure valued solutions reduce to the standard notion of weak solutions of (1.1) if $\boldsymbol{\mu} = \delta_{\mathbf{V}(x,t)}$ [7]. However, measure valued solutions are more general than weak solutions and contain information about the oscillatory structure of the solution. Given the extreme difficulty of proving convergence to a weak solution, a more reasonable (and weaker) requirement would be to show convergence of a numerical method to the entropy measure valued solutions of systems of conservation laws. This is a key consistency criterion for the numerical schemes. Furthermore, in recent years, Glimm et al [12] have hypothesized that measure valued solutions are the appropriate notion of solutions for systems of conservation laws in several space dimensions. Several numerical examples [12] show that there is no pointwise convergence of numerical schemes as the mesh is refined. On the other hand, there is convergence of interesting functionals (such as drag in aerodynamic simulations). This convergence in terms of functionals is consistent with the notion of convergence to entropy measure valued solutions [12], and references therein.

1.5 Aims and scope of the current paper

The current paper proposes and analyses an entropy stable spacetime DG method for systems of conservation laws. The schemes are based on the following ingredients:

- The degrees of freedom of the formulation are the entropy variables, as opposed to the conservative variables in standard DG formulations [4].
- The numerical flux function in the DG formulation is the entropy stable flux of Tadmor [27], based on explicit entropy conservative fluxes [11] and appropriate numerical diffusion operators [11]. At lowest order, our scheme reduces to the first-order (implicit) finite volume scheme of [11].
- Shock capturing operators, similar to those proposed in [21] and [1] are introduced to stabilize oscillations around shocks. A novel *pressure scaling* is introduced in order to capture contact discontinuities with sharper resolution.
- Streamline diffusion operators are added for further stabilization of shocks and to enable solution of the resulting nonlinear algebraic equations at each time step.

We obtain (formally) arbitrarily high-order accurate fully discrete entropy stable schemes for a generic systems of conservation laws in several space dimensions on unstructured grids. In addition to the entropy stability, we also prove (under the extra

assumption that the approximate solutions are uniformly bounded in L^∞) that the approximate solutions generated by the shock-capturing streamline diffusion space-time DG method converge to the entropy measure valued solutions of the underlying system of conservation laws. Furthermore, a large number of numerical experiments are presented to illustrate the robustness of the proposed schemes.

The rest of the paper is organized as follows: in Sect. 2, the spacetime DG formulation is presented for a generic system of conservation laws on unstructured grids and the shock capturing and streamline diffusion operators are introduced. The entropy stability analysis is presented in Sect. 3. In Sect. 4, we show that the shock capturing streamline diffusion method converges to an entropy measure valued solution of a system of conservation laws. Implementation details and numerical results are presented in Sects. 5 and 6, respectively.

2 The shock capturing streamline diffusion DG formulation

2.1 The mesh

At the n-th time level t^n, we denote the time step as Δt^n and the update time interval as $I^n = [t^n, t^{n+1})$ and $t^{n+1} - t^n = \Delta t^n$. For simplicity, we assume that the spatial domain $\Omega \subset \mathbb{R}^d$ is polyhedral and divide into a triangulation $\mathcal{T}$ i.e, a set of open convex polyhedra $K \subset \mathbb{R}^d$ with plane faces. Furthermore, we assume mesh regularity [19]. For a generic element (cell) K, we denote

$$\Delta x_K = diam(K),$$
$$\mathcal{N}(K) = \left\{ K' \in \mathcal{T} : K' \neq K \wedge \text{meas}_{d-1}(\overline{K} \cap \overline{K'} > 0 \right\}.$$

The *mesh width* of the triangulation is $\Delta x(\mathcal{T}) = \max_K \Delta x_K$. A generic spacetime element is the prism:

$$K \times I^n.$$

We also assume that there exists a constant C such that $\Delta t^n \leq C \Delta x$ for all time levels n.

2.2 Variational formulation

Given a strictly convex entropy function S, the conservative variables $\mathbf{U}$ and the entropy variables $\mathbf{V}$ are one to one [6]. Consequently, the conservation law (1.1) can be recast in terms of entropy variables as,

$$\mathbf{U}(\mathbf{V})_t + \sum_{k=1}^{d} \mathbf{F}^k(\mathbf{V})_{x_k} = 0, \quad (x, t) \in \Omega \times \mathbb{R}_+, \tag{2.1}$$

Here, we have used the change of variable $\mathbf{U} = \mathbf{U}(\mathbf{V})$ and retained the notation $\mathbf{F}^k(\mathbf{V}) = \mathbf{F}^k(\mathbf{U}(\mathbf{V}))$ for all k for notational convenience. Following [1,28],

we approximate the conservation law (2.1) by a DG method. On a given triangulation $\mathcal{T}$ with mesh width Δx, we seek entropy variables

$$
\begin{aligned}
\mathbf{V}^{\Delta x} \in \mathcal{V}_p &= (\mathbb{P}_p(\Omega \times [0, T]))^m \\
&= \left\{ \mathbf{W} \in (L^1(\Omega \times [0, T]))^m : \mathbf{W}|_{K \times I^n} \text{ is a polynomial of degree } p \text{ in each component} \right\}
\end{aligned}
\tag{2.2}
$$

such that the following quasilinear variational form is satisfied for each $\mathbf{W}^{\Delta x} \in \mathcal{V}_p$:

$$
\mathcal{B}\left(\mathbf{V}^{\Delta x}, \mathbf{W}^{\Delta x}\right) = \mathcal{B}_{DG}\left(\mathbf{V}^{\Delta x}, \mathbf{W}^{\Delta x}\right) + \mathcal{B}_{SD}\left(\mathbf{V}^{\Delta x}, \mathbf{W}^{\Delta x}\right) + \mathcal{B}_{SC}\left(\mathbf{V}^{\Delta x}, \mathbf{W}^{\Delta x}\right) = 0.
\tag{2.3}
$$

We elaborate on each of the three quasilinear forms (nonlinear in the first argument and linear in the second) in the following.

2.3 The DG quasilinear form

The form $\mathcal{B}_{DG}$ is given by,

$$
\begin{aligned}
&\mathcal{B}_{DG}\left(\mathbf{V}^{\Delta x}, \mathbf{W}^{\Delta x}\right) \\
&= -\sum_n \sum_K \int_{I^n} \int_K \left(\langle \mathbf{U}(\mathbf{V}^{\Delta x}), \mathbf{W}_t^{\Delta x} \rangle + \sum_{k=1}^d \langle \mathbf{F}^k(\mathbf{V}^{\Delta x}), \mathbf{W}_{x_k}^{\Delta x} \rangle \right) dx\, dt \\
&\quad + \sum_n \sum_K \int_K \langle \mathbb{U}(\mathbf{V}_{n+1,-}^{\Delta x}, \mathbf{V}_{n+1,+}^{\Delta x}), \mathbf{W}_{n+1,-}^{\Delta x} \rangle dx \\
&\quad - \sum_n \sum_K \int_K \langle \mathbb{U}(\mathbf{V}_{n,-}^{\Delta x}, \mathbf{V}_{n,+}^{\Delta x},), \mathbf{W}_{n,+}^{\Delta x} \rangle dx \\
&\quad + \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left(\sum_{k=1}^d \langle \mathbb{F}^{k,*}\left(\mathbf{V}_{K,-}^{\Delta x}, \mathbf{V}_{K,+}^{\Delta x}\right), \mathbf{W}_{K,-}^{\Delta x} \rangle \right. \\
&\qquad \left. v_{KK'}^k \right) d\sigma(x)\, dt \\
&\quad - \frac{1}{2} \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \langle \mathbf{W}_{K,-}^{\Delta x}, \mathbf{D}(\mathbf{V}_{K,+}^{\Delta x} - \mathbf{V}_{K,-}^{\Delta x}) \rangle d\sigma(x)\, dt \tag{2.4}
\end{aligned}
$$

Here we have employed the notation,

$$
\begin{aligned}
\mathbf{W}_{n,\pm}(x) &= \mathbf{W}(x, t_\pm^n), \\
\partial_{KK'} &= \overline{K} \cap \overline{K'},
\end{aligned}
$$

$v_{KK'}$ = Unit normal for edge KK' pointing outwards from element K.

$$\mathbf{W}_{K,\pm}(x,t) = \lim_{h \to 0} \mathbf{W}(x \pm h\nu, t), \quad \forall x \in \partial_{KK'},$$

$$\mathbf{D} = \mathbf{D}\left(\mathbf{V}_{K,-}^{\Delta x}, \mathbf{V}_{K,+}^{\Delta x}; \nu_{KK'}\right)$$

for all $\mathbf{W} \in \mathcal{V}_p$. We remark that the boundary condition is ignored in the above variational form by considering compactly supported (in the spatial domain) solutions and test functions.

2.3.1 Numerical fluxes

Both the temporal and spatial numerical fluxes, need to be specified in order to complete the DG quasilinear form. In order to obtain casuality (marching) after each time step, we choose the temporal numerical flux to be the *upwind* flux:

$$\mathbb{U}(a,b) = \mathbf{U}(a). \tag{2.5}$$

This ensures that we can use the values at the previous time step in order to compute an update at the time level t^n. A different choice of temporal numerical fluxes will imply that all the degrees of freedom (for all times) are coupled and force us to solve a very large non-linear algebraic system of equations.

The spatial numerical flux consists of the following two components,

2.3.2 Entropy conservative flux

The entropy conservative flux (in the k-th direction) is any flux [27] that satisfies the relation:

$$\langle b - a, \mathbb{F}^{k,*}(a,b) \rangle = \Psi^k(b) - \Psi^k(a). \tag{2.6}$$

Here, $\Psi^k = \langle \mathbf{V}, \mathbf{F}^k \rangle - Q^k$ is the entropy potential. The existence of such fluxes (for any generic conservation law with an entropy framework) was shown by Tadmor in [27]. More recently, explicit expressions of entropy conservative fluxes for specific systems of interest like the shallow water equations [9] and Euler equations [18] have been obtained.

2.3.3 Numerical diffusion operators

Following [9,11,28], we choose the numerical diffusion operator as,

$$\mathbf{D}(a,b;\nu) = \mathbf{R}_\nu \mathbf{P}(\Lambda_\nu(\cdot); a, b)\mathbf{R}_\nu^\top. \tag{2.7}$$

Here, Λ_ν, $\mathbf{R}_\nu$ are the eigenvalue and eigenvector matrices of the Jacobian $\partial_{\mathbf{U}}(\langle \mathbf{F}, \nu \rangle)$ in the normal direction ν. $\mathbf{R}_\nu$ is evaluated at an averaged state, e.g. $(a+b)/2$, and scaled such that $\mathbf{R}_\nu \mathbf{R}_\nu^\top = \mathbf{U}_\mathbf{V}$. $\mathbf{P}$ is a non-negative matrix function. Examples of $\mathbf{P}$ include $\mathbf{P}(\Lambda_\nu(\cdot); a, b) = |\Lambda_\nu(\frac{a+b}{2})|$ which leads to a Roe type scheme and $\mathbf{P}(\Lambda_\nu(\cdot); a, b) =$

$\max\{\lambda_{max}(a; \nu), \lambda_{max}(b; \nu)\}\mathbf{ID}$, which leads to a Rusanov type scheme [11], where $\lambda_{max}(\mathbf{U}; \nu)$ is the maximal wave speed in direction of ν, i.e. $\lambda_{max}(\mathbf{U}; \nu)$ is the spectral radius of $\Lambda_\nu(\mathbf{U})$.

2.4 Streamline diffusion operator

As the subsequent analysis will demonstrate, there is no numerical diffusion in the interior of the space-time element $K \times I^n$. In order to suppress the resulting unphysical oscillations near shocks, we choose the following streamline diffusion operator,

$$
\mathcal{B}_{SD}(\mathbf{V}^{\Delta x}, \mathbf{W}^{\Delta x})
$$
$$
= \sum_n \sum_K \int_{I^n} \int_K \left\langle \left(\mathbf{U_V}(\mathbf{V}^{\Delta x})\mathbf{W}_t^{\Delta x} + \sum_{k=1}^{d} \mathbf{F}_\mathbf{V}^k(\mathbf{V}^{\Delta x})\mathbf{W}_{x_k}^{\Delta x} \right), \mathbf{D}^{SD} \operatorname{Res} \right\rangle dx dt \quad (2.8)
$$

with intra-element residual:

$$
\operatorname{Res} = \mathbf{U}(\mathbf{V}^{\Delta x})_t + \sum_{k=1}^{d} \mathbf{F}^k(\mathbf{V}^{\Delta x})_{x_k}, \quad (2.9)
$$

and the scaling matrix is chosen as

$$
\mathbf{D}^{SD} = C^{SD}\Delta x \mathbf{ID}, \quad (2.10)
$$

for any positive constant C^{SD}. Note that the intra-element residual is well defined as we are taking first-derivatives of a polynomial function.

2.5 Shock capturing operator

The streamline diffusion operator adds numerical diffusion in the direction of the streamlines. However, we need further numerical diffusion in order to reduce possible oscillations at shocks. We use the following shock capturing operator (similar to Barth [1]):

$$
\mathcal{B}_{SC}(\mathbf{V}^{\Delta x}, \mathbf{W}^{\Delta x}) = \sum_n \sum_K \int_{I^n} \int_K D_{n,K}^{SC} \left(\langle \mathbf{W}_t^{\Delta x}, \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K})\mathbf{V}_t^{\Delta x} \rangle \right.
$$
$$
\left. + \sum_{k=1}^{d} \langle \mathbf{W}_{x_k}^{\Delta x}, \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K})\mathbf{V}_{x_k}^{\Delta x} \rangle \right) dx dt, \quad (2.11a)
$$

with

$$
\tilde{\mathbf{V}}_{n,K} = \frac{1}{\operatorname{meas}(I^n \times K)} \int_{I^n} \int_K \mathbf{V}^{\Delta x}(x, t) dx dt.
$$

being the cell average and the scaling factor,

$$D_{n,K}^{SC} = \frac{(\Delta x)^{1-\alpha}C^{SC}\overline{\mathrm{Res}}_{n,K} + (\Delta x)^{\frac{1}{2}-\alpha}\bar{C}^{SC}\overline{\mathrm{BRes}}_{n,K}}{\sqrt{\int_{I^n}\int_K(\langle \mathbf{V}_t^{\Delta x}, \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K})\mathbf{V}_t^{\Delta x}\rangle + \sum_{k=1}^d \langle \mathbf{V}_{x_k}^{\Delta x}, \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K})\mathbf{V}_{x_k}^{\Delta x}\rangle)dxdt + \Delta x^\theta}},$$

(2.11b)

with $\theta \geq \alpha + d/2$ and

$$\overline{\mathrm{Res}}_{n,K} = \sqrt{\int_{I^n}\int_K \langle \mathrm{Res}, \mathbf{U_V}^{-1}(\mathbf{V}^{\Delta x})\,\mathrm{Res}\rangle dxdt}.$$

(2.11c)

$$\overline{\mathrm{BRes}}_{n,K} = \left(\int_K \|\mathbf{U}(\mathbf{V}_{n,-}^{\Delta x}) - \mathbf{U}(\mathbf{V}_{n,+}^{\Delta x})\|^2 dx\right.$$

$$+ \sum_{K'\in\mathcal{N}(K)}\int_{I^n}\int_{\partial_{KK'}}\left(\sum_{k=1}^d \|(\mathbb{F}^{k,*}(\mathbf{V}_{K,-}^{\Delta x}, \mathbf{V}_{K,+}^{\Delta x}) - \mathbf{F}^k(\mathbf{V}_{K,-}^{\Delta x}))\nu_{KK'}^k\|^2\right.$$

$$\left.\left. + \left\|\frac{1}{2}\mathbf{D}(\mathbf{V}_{K,+}^{\Delta x} - \mathbf{V}_{K,-}^{\Delta x})\right\|^2\right)d\sigma(x)dt\right)^{\frac{1}{2}}$$

(2.11d)

Here, $C^{SC}, \bar{C}^{SC}$ are positive constants.

3 Entropy stability for nonlinear systems

The design of the streamline diffusion (SD)-shock capturing (SC)-discontinuous Galerkin(DG) scheme (2.3) is motivated by the consideration that it has to be entropy stable for a generic nonlinear system of conservation laws, equipped with an entropy formulation. We have the following theorem on entropy stability:

Theorem 3.1 *Consider the system of conservation laws* (1.1) *with strictly convex entropy function S and entropy flux functions $Q_{(1\leq k\leq d)}^k$. For simplicity, assume that the exact and approximate solutions have compact support inside the spatial domain Ω. Let the final time be denoted by t_-^N. Then, the streamline diffusion-shock capturing-Discontinuous Galerkin scheme* (2.3) *approximating* (1.1) *has the following properties:*

(i) *The scheme* (2.3) *is conservative i.e, the approximate solutions $\mathbf{U}^{\Delta x} = \mathbf{U}(\mathbf{V}^{\Delta x})$ satisfy*

$$\int_\Omega \mathbf{U}(\mathbf{V}^{\Delta x}(x, t_-^N))dx = \int_\Omega \mathbf{U}(\mathbf{V}^{\Delta x}(x, t_-^0))dx.$$

(3.1)

(ii) *The scheme* (2.3) *is entropy stable i.e, the approximate solutions satisfy,*

$$\int_\Omega S\left(\mathbf{U}^*(t_-^0)\right) dx \le \int_\Omega S\left(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_-^N))\right) dx \le \int_\Omega S\left(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_-^0))\right) dx,$$

(3.2)

with $\mathbf{U}^*$ *being the domain average:*

$$\mathbf{U}^*(t_-^0) = \frac{1}{meas(\Omega)} \int_\Omega \mathbf{U}(\mathbf{V}(x, t_-^0))dx.$$

(iii) *We obtain the following weak "BV" estimate:*

$$\sum_n \sum_K \int_K \|\mathbf{V}_{n,-}^{\Delta x} - \mathbf{V}_{n,+}^{\Delta x}\|^2 dx$$

$$+ \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \langle \mathbf{V}_{K,+}^{\Delta x} - \mathbf{V}_{K,-}^{\Delta x}, \mathbf{D}(\mathbf{V}_{K,+}^{\Delta x} - \mathbf{V}_{K,-}^{\Delta x}) \rangle d\sigma(x)dt$$

$$+ \Delta x \sum_n \sum_K \int_{I^n} \int_K \|\mathbf{U}(\mathbf{V}^{\Delta x})_t + \sum_{k=1}^d \mathbf{F}^k(\mathbf{V}^{\Delta x}))_{x_k}\|^2 dxdt$$

$$+ (\Delta x)^{1-\alpha} \sum_n \sum_K \overline{\mathrm{Res}}_{n,K} \left(\int_{I^n} \int_K \|\nabla_{xt} \mathbf{V}^{\Delta x}\|^2 dxdt \right)^{\frac{1}{2}}$$

$$+ (\Delta x)^{\frac{1}{2}-\alpha} \sum_n \sum_K \overline{\mathrm{BRes}}_{n,K} \left(\int_{I^n} \int_K \|\nabla_{xt} \mathbf{V}^{\Delta x}\|^2 dxdt \right)^{\frac{1}{2}} \le C.$$

(3.3)

For some constant C, depending on the initial data and with the spacetime gradient defined by,

$$\nabla_{xt} \mathbf{V}^{\Delta x} = \left(\mathbf{V}_t^{\Delta x}, \mathbf{V}_{x_1}^{\Delta x}, \mathbf{V}_{x_2}^{\Delta x}, \cdots, \mathbf{V}_{x_d}^{\Delta x} \right).$$

(3.4)

Proof The proof of conservation (property (i)) (3.1) is a straightforward consequence of setting the test function $\mathbf{W}^{\Delta x} = 1$ and the definition of the numerical flux in (2.4).

$\square$

To prove entropy stability, we proceed to show a series of claims.

Claim 1 The streamline diffusion operator (2.8) is positive i.e,

$$\mathcal{B}_{SD}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) \ge 0.$$

(3.5)

As, $\mathbf{V}^{\Delta x} \in \mathcal{V}_p$, it is an admissible test function in the quasilinear form $\mathcal{B}_{SD}$. Setting $\mathbf{W}^{\Delta x} = \mathbf{V}^{\Delta x}$ in (2.8), we obtain

$$
\begin{aligned}
\mathcal{B}_{SD}\left(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}\right) &= \sum_n \sum_K \int_{I^n} \int_K \left\langle \left(\mathbf{U_V}(\mathbf{V}^{\Delta x})\mathbf{V}_t^{\Delta x} + \sum_{k=1}^d \mathbf{F}_\mathbf{V}^k(\mathbf{V}^{\Delta x})\mathbf{V}_{x_k}^{\Delta x} \right), \mathbf{D}^{SD}\,\mathrm{Res} \right\rangle dx\,dt \\
&= \sum_n \sum_K \int_{I^n} \int_K \left\langle \left(\mathbf{U}(\mathbf{V}^{\Delta x})_t + \sum_{k=1}^d \mathbf{F}^k(\mathbf{V}^{\Delta x}))_{x_k} \right), \mathbf{D}^{SD}\,\mathrm{Res} \right\rangle dx\,dt \\
&= \sum_n \sum_K \int_{I^n} \int_K \left\langle \mathrm{Res}, \mathbf{D}^{SD}\mathrm{Res} \right\rangle dx\,dt \quad \text{from defn (2.9)}, \\
&= C^{SD} \Delta x \sum_n \sum_K \int_{I^n} \int_K \|\mathrm{Res}\|^2 dx\,dt \\
&= C^{SD} \Delta x \sum_n \sum_K \int_{I^n} \int_K \left\| \mathbf{U}(\mathbf{V}^{\Delta x})_t + \sum_{k=1}^d \mathbf{F}^k(\mathbf{V}^{\Delta x}))_{x_k} \right\|^2 dx\,dt \\
&\geq 0.
\end{aligned}
\tag{3.6}
$$

Claim 2 The shock capturing operator (2.11a, 2.11b, 2.11c, 2.11d) is positive i.e:

$$
\mathcal{B}_{SC}\left(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}\right) \geq 0.
\tag{3.7}
$$

First, we observe that the strict convexity of the entropy function S implies that the matrices $\mathbf{U_V}$ and $\mathbf{U_V}^{-1}$ are strictly positive definite. This implies that the term $D_{n,K}^{SC} \geq 0$.

We set as test function, $\mathbf{W}^{\Delta x} = \mathbf{V}^{\Delta x}$ in (2.11a) and obtain,

$$
\begin{aligned}
\mathcal{B}_{SC}\left(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}\right) &= \sum_n \sum_K \int_{I^n} \int_K D_{n,K}^{SC} \left(\langle \mathbf{V}_t^{\Delta x}, \mathbf{U_V}(\tilde{\mathbf{V}}^{\Delta x})\mathbf{V}_t^{\Delta x} \rangle \right. \\
&\quad \left. + \sum_{k=1}^d \langle \mathbf{V}_{x_k}^{\Delta x}, \mathbf{U_V}(\tilde{\mathbf{V}}^{\Delta x})\mathbf{V}_{x_k}^{\Delta x} \rangle \right) dx\,dt, \\
&\geq \sum_n \sum_K \lambda_1 D_{n,K}^{SC} \int_{I^n} \int_K \|\nabla_{xt} \mathbf{V}^{\Delta x}\|^2 dx\,dt \\
&\geq 0.
\end{aligned}
\tag{3.8}
$$

Here, λ_1 is the smallest eigenvalue of the positive definite matrix $\mathbf{U_V}$.

Claim 3 Define a part of the DG form $\mathcal{B}_{DG}$ (2.4) as,

$$
\begin{aligned}
\mathcal{B}_{DG}^s\left(\mathbf{V}^{\Delta x}, \mathbf{W}^{\Delta x}\right) &= -\sum_n \sum_K \int_{I^n} \int_K \sum_{k=1}^d \left\langle \mathbf{F}^k(\mathbf{V}^{\Delta x}), \mathbf{W}_{x_k}^{\Delta x} \right\rangle dx\,dt \\
&\quad + \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left(\sum_{k=1}^d \left\langle \mathbb{F}^{k,*}\left(\mathbf{V}_{K,-}^{\Delta x}, \mathbf{V}_{K,+}^{\Delta x}\right), \mathbf{W}_{K,-}^{\Delta x} \right\rangle \nu_{KK'}^k \right) d\sigma(x)\,dt \\
&\quad - \frac{1}{2} \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \langle \mathbf{W}_{K,-}^{\Delta x}, \mathbf{D}\left(\mathbf{V}_{K,+}^{\Delta x} - \mathbf{V}_{K,-}^{\Delta x}\right) \rangle d\sigma(x)\,dt
\end{aligned}
\tag{3.9}
$$

Then, we have the following estimate:

$$\mathcal{B}^s_{DG}\left(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}\right) \geq 0. \tag{3.10}$$

From the definition of the entropy potential Ψ^k, we obtain that for all $1 \leq k \leq d$,

$$
\begin{aligned}
\Psi^k_{x_k} &= \left(\left\langle \mathbf{V}, \mathbf{F}^k \right\rangle - Q^k\right)_{x_k} \\
&= \left\langle \mathbf{V}_{x_k}, \mathbf{F}^k \right\rangle + \langle \mathbf{V}, \mathbf{F}^k_{x_k} \rangle - (Q^k)_{x_k} \\
&= \left\langle \mathbf{V}_{x_k}, \mathbf{F}^k \right\rangle \quad (\text{from definition of } Q^k).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_n \sum_K \int_{I^n} \int_K \sum_{k=1}^d \left\langle \mathbf{F}^k(\mathbf{V}^{\Delta x}), \mathbf{V}^{\Delta x}_{x_k} \right\rangle dx dt &= \sum_n \sum_K \int_{I^n} \int_K \sum_{k=1}^d \Psi^k\left(\mathbf{V}^{\Delta x}\right)_{x_k} dx dt \\
&= \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \sum_{k=1}^d \Psi^k\left(\mathbf{V}^{\Delta x}_{K,-}\right) v^k_{KK'} d\sigma(x) dt.
\end{aligned}
$$

Using the above identities, we obtain,

$$
\begin{aligned}
\mathcal{B}^s_{DG}\left(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}\right) &= -\sum_n \sum_K \int_{I^n} \int_K \sum_{k=1}^d \left\langle \mathbf{F}^k\left(\mathbf{V}^{\Delta x}\right), \mathbf{V}^{\Delta x}_{x_k} \right\rangle dx dt \\
&\quad + \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left(\sum_{k=1}^d \langle \mathbb{F}^{k,*}\left(\mathbf{V}^{\Delta x}_{K,-}, \mathbf{V}^{\Delta x}_{K,+}\right), \mathbf{V}^{\Delta x}_{K,-} \rangle \right. \\
&\qquad \left. v^k_{KK'} \right) d\sigma(x) dt. \\
&\quad - \frac{1}{2} \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left\langle \mathbf{V}^{\Delta x}_{K,-}, \mathbf{D}(\mathbf{V}^{\Delta x}_{K,+} - \mathbf{V}^{\Delta x}_{K,-}) \right\rangle d\sigma(x) dt \\
&= \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \\
&\qquad \left(\sum_{k=1}^d (\langle \mathbb{F}^{k,*}(\mathbf{V}^{\Delta x}_{K,-}, \mathbf{V}^{\Delta x}_{K,+}), \mathbf{V}^{\Delta x}_{K,-} \rangle - \Psi^k(\mathbf{V}^{\Delta x}_{K,-})) v^k_{KK'} \right) d\sigma(x) dt \\
&\quad - \frac{1}{2} \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left\langle \mathbf{V}^{\Delta x}_{K,-}, \mathbf{D}(\mathbf{V}^{\Delta x}_{K,+} - \mathbf{V}^{\Delta x}_{K,-}) \right\rangle d\sigma(x) dt
\end{aligned}
$$

$$= \frac{1}{2} \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left(\sum_{k=1}^{d} (\langle \mathbb{F}^{k,*}(\mathbf{V}_{K,-}^{\Delta x}, \mathbf{V}_{K,+}^{\Delta x}), \mathbf{V}_{K,-}^{\Delta x} \rangle \right.$$

$$\left. - \Psi^k(\mathbf{V}_{K,-}^{\Delta x})) \nu_{KK'}^k \right) d\sigma(x) dt$$

$$+ \frac{1}{2} \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left(\sum_{k=1}^{d} (\langle \mathbb{F}^{k,*}(\mathbf{V}_{K,-}^{\Delta x}, \mathbf{V}_{K,+}^{\Delta x}), \mathbf{V}_{K,-}^{\Delta x} \rangle \right.$$

$$\left. - \Psi^k(\mathbf{V}_{K,-}^{\Delta x})) \nu_{KK'}^k \right) d\sigma(x) dt$$

$$- \frac{1}{4} \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \langle \mathbf{V}_{K,-}^{\Delta x}, \mathbf{D}(\mathbf{V}_{K,+}^{\Delta x} - \mathbf{V}_{K,-}^{\Delta x}) \rangle d\sigma(x) dt$$

$$- \frac{1}{4} \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \langle \mathbf{V}_{K,-}^{\Delta x}, \mathbf{D}(\mathbf{V}_{K,+}^{\Delta x} - \mathbf{V}_{K,-}^{\Delta x}) \rangle d\sigma(x) dt$$

$$(3.11)$$

Changing the orientation of the unit normal on the face KK' and rewriting the arguments of the sums using the fact that the approximate solutions have compact support in Ω, the above expression reduces to

$$\mathcal{B}_{DG}^s \left(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x} \right) = -\frac{1}{2} \sum_{n,K} \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}}$$

$$\times \left(\sum_{k=1}^{d} \underbrace{(\langle \mathbb{F}^{k,*}(\mathbf{V}_{K,-}^{\Delta x}, \mathbf{V}_{K,+}^{\Delta x}), \mathbf{V}_{K,+}^{\Delta x} - \mathbf{V}_{K,-}^{\Delta x} \rangle - (\Psi^k(\mathbf{V}_{K,+}^{\Delta x}) - \Psi^k(\mathbf{V}_{K,-}^{\Delta x})))}_{=0 \text{ from defn entropy conservative flux(2.6)}} \nu_{KK'}^k \right) d\sigma(x) dt$$

$$+ \frac{1}{4} \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \langle \mathbf{V}_{K,+}^{\Delta x} - \mathbf{V}_{K,-}^{\Delta x}, \mathbf{D}(\mathbf{V}_{K,+}^{\Delta x} - \mathbf{V}_{K,-}^{\Delta x}) \rangle d\sigma(x) dt$$

$$= \frac{1}{4} \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \langle \mathbf{V}_{K,+}^{\Delta x} - \mathbf{V}_{K,-}^{\Delta x}, \mathbf{D}(\mathbf{V}_{K,+}^{\Delta x} - \mathbf{V}_{K,-}^{\Delta x}) \rangle d\sigma(x) dt$$

$$0 \quad (\text{from}(2.7)). \tag{3.12}$$

Claim 4 Defining a part of the DG form $\mathcal{B}_{DG}$ (2.4) as

$$\mathcal{B}_{DG}^t \left(\mathbf{V}^{\Delta x}, \mathbf{W}^{\Delta x} \right) = -\sum_n \sum_K \int_{I^n} \int_K \langle \mathbf{U}(\mathbf{V}^{\Delta x}), \mathbf{W}_t^{\Delta x} \rangle dx dt$$

$$+ \sum_n \sum_K \int_K \langle \mathbf{U}(\mathbf{V}_{n+1,-}^{\Delta x}), \mathbf{W}_{n+1,-}^{\Delta x} \rangle dx - \sum_n \sum_K \int_K \langle \mathbf{U}(\mathbf{V}_{n,-}^{\Delta x}), \mathbf{W}_{n,+}^{\Delta x} \rangle dx$$

$$(3.13)$$

(here, we have directly assumed that the temporal numerical flux $\mathbb{U}$ is assumed to be "upwind" (2.5)). The above defined part of the DG form satisfies the estimate,

$$\mathcal{B}_{DG}^t(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) \geq \int_{\Omega} S\Big(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_-^N))\Big) dx - \int_{\Omega} S\Big(\mathbf{U}\big(\mathbf{V}^{\Delta x}(x, t_-^0)\big)\Big) dx \quad (3.14)$$

Setting $\mathbf{W}^{\Delta x} = \mathbf{V}^{\Delta x}$ in (3.13), we obtain,

$$\begin{aligned}
\mathcal{B}_{DG}^t(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) &= -\sum_n \sum_K \int_{I^n} \int_K \Big\langle \mathbf{U}(\mathbf{V}^{\Delta x}), \mathbf{V}_t^{\Delta x} \Big\rangle dx dt \\
&\quad + \sum_n \sum_K \int_K \Big\langle \mathbf{U}(\mathbf{V}_{n+1,-}^{\Delta x}), \mathbf{V}_{n+1,-}^{\Delta x} \Big\rangle dx \\
&\quad - \sum_n \sum_K \int_K \Big\langle \mathbf{U}(\mathbf{V}_{n,-}^{\Delta x}), \mathbf{V}_{n,+}^{\Delta x} \Big\rangle dx \\
&= \sum_n \sum_K \int_{I^n} \int_K \Big\langle \mathbf{U}_t(\mathbf{V}^{\Delta x}), \mathbf{V}^{\Delta x} \Big\rangle dx dt \quad \text{(integrating by parts)} \\
&\quad - \sum_n \sum_K \int_K \Big\langle \mathbf{U}(\mathbf{V}_{n+1,-}^{\Delta x}), \mathbf{V}_{n+1,-}^{\Delta x} \Big\rangle dx \\
&\quad + \sum_n \sum_K \int_K \Big\langle \mathbf{U}(\mathbf{V}_{n,+}^{\Delta x}), \mathbf{V}_{n,+}^{\Delta x} \Big\rangle dx \\
&\quad + \sum_n \sum_K \int_K \Big\langle \mathbf{U}(\mathbf{V}_{n+1,-}^{\Delta x}), \mathbf{V}_{n+1,-}^{\Delta x} \Big\rangle dx \\
&\quad - \sum_n \sum_K \int_K \Big\langle \mathbf{U}(\mathbf{V}_{n,-}^{\Delta x}), \mathbf{V}_{n,+}^{\Delta x} \Big\rangle dx \\
&= \sum_n \sum_K \int_{I^n} \int_K S(\mathbf{U}(\mathbf{V}^{\Delta x}))_t \, dx dt \quad \text{(Definition of entropy function)} \\
&\quad - \sum_n \sum_K \int_K \Big\langle (\mathbf{U}(\mathbf{V}_{n,-}^{\Delta x}) - \mathbf{U}(\mathbf{V}_{n,+}^{\Delta x})), \mathbf{V}_{n,+}^{\Delta x} \Big\rangle dx \\
&= \sum_n \sum_K \int_K \big(S(\mathbf{U}(\mathbf{V}_{n+1,-}^{\Delta x})) - S(\mathbf{U}(\mathbf{V}_{n,-}^{\Delta x})) \big) dx \\
&\quad + \sum_n \sum_K \int_K \big(S(\mathbf{U}(\mathbf{V}_{n,-}^{\Delta x})) - S(\mathbf{U}(\mathbf{V}_{n,+}^{\Delta x})) \big) dx \\
&\quad - \sum_n \sum_K \int_K \Big\langle (\mathbf{U}(\mathbf{V}_{n,-}^{\Delta x}) - \mathbf{U}(\mathbf{V}_{n,+}^{\Delta x})), \mathbf{V}_{n,+}^{\Delta x} \Big\rangle dx
\end{aligned}$$

$$= \int_\Omega S\left(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_-^N))\right)dx - \int_\Omega S(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_-^0)))\,dx$$

$$+ \sum_n \sum_K \int_K \int_0^1 \langle (\mathbf{V}_{n,-}^{\Delta x} - \mathbf{V}_{n,+}^{\Delta x}), \mathbf{U}_{\mathbf{V}}(\theta)\,(\mathbf{V}_{n,-}^{\Delta x} - \mathbf{V}_{n,+}^{\Delta x})\rangle\,d\theta$$

(change of variables: $\mathbf{V}(\theta) = \theta\mathbf{V}_{n,-}^{\Delta x} + (1 - \theta)\mathbf{V}_{n,+}^{\Delta x}$)

$$\geq \int_\Omega S\left(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_-^N))\right)dx - \int_\Omega S\left(\mathbf{U}\left(\mathbf{V}^{\Delta x}(x, t_-^0)\right)\right)dx$$

$$+ \sum_n \sum_K \lambda_1 \int_K \|\mathbf{V}_{n,-}^{\Delta x} - \mathbf{V}_{n,+}^{\Delta x}\|^2 dx$$

$$\geq \int_\Omega S\left(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_-^N))\right)dx - \int_\Omega S\left(\mathbf{U}\left(\mathbf{V}^{\Delta x}(x, t_-^0)\right)\right)dx \quad (3.15)$$

Now combining the four claims, we obtain

$$\mathcal{B}_{DG}\left(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}\right) + \mathcal{B}_{SD}\left(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}\right) + \mathcal{B}_{SC}\left(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}\right) = 0,$$

$$\Rightarrow \quad \mathcal{B}_{DG}^t\left(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}\right) + \mathcal{B}_{DG}^s\left(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}\right) + \mathcal{B}_{SD}\left(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}\right) + \mathcal{B}_{SC}\left(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}\right) = 0,$$

$$\Rightarrow \quad \int_\Omega S\left(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_-^N))\right)dx - \int_\Omega S\left(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_-^0))\right)dx \leq 0,$$

$$\Rightarrow \quad \int_\Omega S\left(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_-^N))\right)dx \leq \int_\Omega S\left(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_-^0))\right)dx.$$

Thus, the upper bound in the entropy estimate (3.2).

To prove the lower bound on entropy in estimate (3.2), we follow Barth [2] and define the domain average,

$$\mathbf{U}^*(t_-^n) = \frac{1}{\text{meas}(\Omega)} \int_\Omega \mathbf{U}\left(\mathbf{V}^{\Delta x}(x, t_-^n)\right)dx. \quad (3.16)$$

From conservation, we know that $\mathbf{U}^*(t_-^n) = \mathbf{U}^*(t_-^0)$.

For any given time level, we have

$$S(\mathbf{U}) = S(\mathbf{U}^*) + \langle \mathbf{V}(\mathbf{U}^*), (\mathbf{U} - \mathbf{U}^*)\rangle + \int_0^1 \langle (\mathbf{U} - \mathbf{U}^*), S_{\mathbf{U}\mathbf{U}}(\mathbf{U}(\theta))(\mathbf{U} - \mathbf{U}^*)\rangle\,d\theta,$$

with change of variables $\mathbf{U}(\theta) = \theta\mathbf{U} + (1 - \theta)\mathbf{U}^*$. Integrating the above equation over space and using the definition of domain average (3.16) and the strict convexity of the entropy function, we obtain,

$$\int_\Omega S(\mathbf{U}^*)dx \le \int_\Omega S(\mathbf{U})dx$$

for any time level.

As a consequence of conservation, we obtain,

$$\int_\Omega S\left(\mathbf{U}^*(t^0_-)\right)dx = \int_\Omega S(\mathbf{U}^*(t^N_-))dx \le \int_\Omega S(\mathbf{U}\left(\mathbf{V}^{\Delta x}(x, t^N_-)\right))dx,$$

thus obtaining the lower bound in the entropy estimate (3.2).

Combining the above claims of positivity and using the above estimate results in the following weak BV estimate,

$$\sum_n \sum_K \int_K \|\mathbf{V}^{\Delta x}_{n,-} - \mathbf{V}^{\Delta x}_{n,+}\|^2 dx$$

$$+ \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \langle \mathbf{V}^{\Delta x}_{K,+} - \mathbf{V}^{\Delta x}_{K,-}, \mathbf{D}\left(\mathbf{V}^{\Delta x}_{K,+} - \mathbf{V}^{\Delta x}_{K,-}\right)\rangle d\sigma(x)dt$$

$$+ \Delta x \sum_n \sum_K \int_{I^n} \int_K \|\mathbf{U}(\mathbf{V}^{\Delta x})_t + \sum_{k=1}^d \mathbf{F}^k(\mathbf{V}^{\Delta x}))_{x_k}\|^2 dxdt$$

$$+ \sum_n \sum_K \lambda_1 D^{SC}_{n,K} \int_{I^n} \int_K \|\nabla_{xt}\mathbf{V}\|^2 dxdt \le C. \tag{3.17}$$

The weak BV estimate (3.3) follows from the following straightforward estimate,

$$(\Delta x)^{1-\alpha} \sum_n \sum_K \overline{\mathrm{Res}}_{n,K} \left(\int_{I^n} \int_K \|\nabla_{xt}\mathbf{V}^{\Delta x}\|^2 dxdt\right)^{\frac{1}{2}}$$

$$+ (\Delta x)^{\frac{1}{2}-\alpha} \sum_n \sum_K \overline{\mathrm{BRes}}_{n,K} \left(\int_{I^n} \int_K \|\nabla_{xt}\mathbf{V}^{\Delta x}\|^2 dxdt\right)^{\frac{1}{2}}$$

$$\le C \sum_n \sum_K D^{SC}_{n,K} \int_{I^n} \int_K \|\nabla_{xt}\mathbf{V}^{\Delta x}\|^2 dxdt \tag{3.18}$$

that is a consequence of the definitions of $D^{SC}_{n,K}$, θ and the estimate (3.17).

Remark 3.2 The lowest order version of the streamline diffusion-shock capturing-DG scheme (2.3) is given by considering the test functions in the space $\mathcal{V}_0$ and reduces to a generalization (with implicit time stepping) of the first-order finite volume scheme proposed by Tadmor in [27]. Thus, the scheme (2.3) can be considered as a finite element fully discrete generalization of well-known entropy stable finite volume schemes.

4 Convergence analysis for systems of conservation laws

The ultimate goal for the numerical analysis of the streamline diffusion-shock capturing DG scheme (2.3) would be to show that the approximate solution converge to the entropy solution of the multi-dimensional system (1.1). However, such a result is currently beyond the reach of analysis as there are no well-posedness results for the continuous problem (even for small initial data in several space dimensions). Consequently, the analysis of numerical schemes for conservation laws has focussed on two special cases:

(i) Scalar conservation laws i.e, (1.1) with $m = 1$.
(ii) Linear symmetrizable systems i.e, conservation laws of the form,

$$\mathbf{U}_t + \sum_{k=1}^{d} A^k \mathbf{U}_{x_k} = 0, \quad (x, t) \in \Omega \times \mathbb{R}_+. \tag{4.1}$$

Here, $A^k \in \mathbb{R}^{m \times m}$ are constant matrices (for simplicity). Furthermore, we assume that there exists $B \in \mathbb{R}^{m \times m}$ such that

(a) B is symmetric, (strictly) positive definite.
(b) For all $1 \leq k \leq d$, the matrix BA^k is symmetric.

If such a "symmetrizer" B exists, then the linear system (4.1) is termed a symmetrizable or Friedrichs system. It is easy to check that the symmetrizable system (4.1) is equipped with the following entropy formulation:

$$S(\mathbf{U}) = \frac{1}{2}\langle B\mathbf{U}, \mathbf{U}\rangle, \quad Q^k(U) = \frac{1}{2}\langle \mathbf{U}, BA^k\mathbf{U}\rangle, \quad \mathbf{V} = B\mathbf{U}, \quad \Psi^k(\mathbf{U}) = Q^k(\mathbf{U}). \tag{4.2}$$

A robust numerical scheme should converge to the entropy solutions of scalar conservation laws and the weak solutions of the linear symmetrizable system. We will show that our numerical scheme (2.3) does indeed converge in both the above cases.

However, we would like to prove some results for the general case of a nonlinear system in several space dimensions (in addition to entropy stability). Given the discussion in the introduction about the suitability of entropy measure valued solutions (1.6), (1.7) as the appropriate solution concept in this context, the main theoretical results of this paper are to show that the numerical scheme (2.3) converge to a entropy measure valued solution for a nonlinear system of conservation laws (1.1). Furthermore, we will show that in the special cases of scalar conservation laws as well as linear symmetrizable systems, convergence to measure valued solutions (together with consistency with initial data) automatically implies convergence to entropy solutions. We start with the following convergence theorem,

Theorem 4.1 *Let* $\mathbf{U}^{\Delta x} = \mathbf{U}(\mathbf{V}^{\Delta x})$ *be the approximate solutions of the system* (1.1) *generated by the streamline diffusion-shock capturing DG scheme* (2.3). *Under the assumption that the approximation solutions satisfy the uniform* L^∞ *bound,*

$$\|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega \times \mathbb{R}_+)} \leq C, \tag{4.3}$$

the approximate solutions converge to a measure valued solution (1.6) *of the conservation law* (1.1).

Proof For simplicity, we will only consider the case $p \geq 1$. The $p = 0$ case can be proved using a Lax–Wendroff type argument [13].

To show convergence of the approximate solutions to a measure valued solution of (1.1), we consider any compactly supported test function $\boldsymbol{\varphi} \in \left(C_c^\infty(\Omega \times (0, \infty))\right)^m$ and denote its local H^1 projection into the space $(\mathbb{P}_p)^m$ as $\boldsymbol{\varphi}^{\Delta x}$ i.e,

$$\boldsymbol{\varphi}^{\Delta x} = \Pi^{\Delta x}(\boldsymbol{\varphi}), \tag{4.4}$$

with $\Pi^{\Delta x}|_{K \times I^n} : (L^2)^m \mapsto (\mathbb{P}_p(K \times I^n))^m$ satisfies

$$\int\limits_{I^n} \int\limits_K \left\langle \nabla_{xt}(\Pi^{\Delta x}(\boldsymbol{\varphi})), \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K}) \nabla_{xt} \mathbf{W} \right\rangle dx dt = \int\limits_{I^n} \int\limits_K \left\langle \nabla_{xt}(\boldsymbol{\varphi}), \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K}) \nabla_{xt} \mathbf{W} \right\rangle dx dt,$$

$$\forall \mathbf{W} \in (\mathbb{P}_p(K \times I^n))^m$$

$$\int\limits_{I^n} \int\limits_K \Pi^{\Delta x}(\boldsymbol{\varphi}) dx dt = \int\limits_{I^n} \int\limits_K \boldsymbol{\varphi} dx dt \tag{4.5a}$$

Note that the scaling $\mathbf{U_V}$ is a constant matrix as it is evaluated at the (spacetime) cell average $\tilde{\mathbf{V}}_{n,K}$. Hence, this projection operator for the infinitely smooth function $\boldsymbol{\varphi}$ satisfies the following stability and approximation properties (see [19] or other references therein),

$$\|\nabla_{xt}\boldsymbol{\varphi}^{\Delta x}\|_{L^2(K \times I^n)} \leq C \|\nabla_{xt}\boldsymbol{\varphi}\|_{L^2(K \times I^n)}$$
$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\Delta x}\|_{L^2(K \times I^n)} \leq C \Delta x \|\nabla_{xt}\boldsymbol{\varphi}\|_{L^2(K \times I^n)}. \tag{4.5b}$$
$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\Delta x}\|_{L^2(\partial(K \times I^n))} \leq C \Delta x^{\frac{1}{2}} \|\nabla_{xt}\boldsymbol{\varphi}\|_{L^2(K \times I^n)}$$

In the remainder of the paper, we denote a generic positive constant as C. The proof of convergence consists of the following claims:

Claim 1 The streamline diffusion operator defined in (2.8) satisfies,

$$\mathcal{B}_{SD}(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x}) \to 0 \quad \text{as } \Delta x \to 0. \tag{4.6}$$

To prove the claim, we calculate the bilinear form (2.8),

$$\left| \mathcal{B}_{SD}(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x}) \right| = \left| \sum_n \sum_K \int\limits_{I^n} \int\limits_K \left\langle \left(\mathbf{U}(\mathbf{V}^{\Delta x})\boldsymbol{\varphi}_t^{\Delta x} + \sum_{k=1}^d \mathbf{F}^k(\mathbf{V}^{\Delta x})\boldsymbol{\varphi}_{x_k}^{\Delta x} \right), \mathbf{D}^{SD}\text{Res} \right\rangle dx dt \right|$$

$$\leq C \Delta x \|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega \times [0,T])} |\sum_n \sum_K \|\boldsymbol{\varphi}^{\Delta x}\|_{H^1(K \times I^n)} \left(\int\limits_{I^n} \int\limits_K \|\text{Res}\|^2 dx dt \right)^{\frac{1}{2}}$$

$$\leq C \|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega \times [0,T])} \Delta x^{\frac{1}{2}} \left(\sum_{n,K} \|\boldsymbol{\varphi}^{\Delta x}\|^2_{H^1(K \times I^n)} \right)^{\frac{1}{2}} \left(\Delta x \sum_{n,K} \int_{I^n} \int_K \|\text{Res}\|^2 dx dt \right)^{\frac{1}{2}}$$

$$\leq C \|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega \times [0,T])} \Delta x^{\frac{1}{2}} \|\boldsymbol{\varphi}\|_{H^1(\Omega \times [0,T])} \to 0 \quad \text{as } \Delta x \to 0.$$

Here, the last estimate follows the stability of the projection operator and from the weak BV estimate (3.3). *Claim 2* The shock capturing operator defined in (2.11a, 2.11b, 2.11c, 2.11d) satisfies,

$$\mathcal{B}_{SC}\left(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x}\right) \to 0 \quad \text{as } \Delta x \to 0. \tag{4.7}$$

Define

$$\mathcal{B}_{SC,1}(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x}) := \sum_n \sum_K \int_{I^n} \int_K D_{n,K}^{SC,1} \Bigg(\langle \boldsymbol{\varphi}_t^{\Delta x}, \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_t^{\Delta x} \rangle$$

$$+ \sum_{k=1}^d \langle \boldsymbol{\varphi}_{x_k}^{\Delta x}, \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_{x_k}^{\Delta x} \rangle \Bigg) dx dt,$$

with

$$D_{n,K}^{SC,1} = \frac{(\Delta x)^{1-\alpha} C^{SC} \overline{\text{Res}}_{n,K}}{\sqrt{\int_{I^n} \int_K \left(\langle \mathbf{V}_t^{\Delta x}, \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_t^{\Delta x} \rangle + \sum_{k=1}^d \langle \mathbf{V}_{x_k}^{\Delta x}, \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_{x_k}^{\Delta x} \rangle \right) dx dt + \Delta x^\theta}},$$

Using the strict positivity of $\mathbf{U_V}$, we obtain that,

$$D_{n,K}^{SC,1} \leq C \frac{(\Delta x)^{1-\alpha} C^{SC} \overline{\text{Res}}_{n,K}}{\left(\int_K \int_{I^n} \|\nabla_{xt} \mathbf{V}\|^2 dx dt \right)^{\frac{1}{2}} + \Delta x^\theta}.$$

Therefore,

$$|\mathcal{B}_{SC,1}(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x})| \leq C(\Delta x)^{1-\alpha} \|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega \times [0,T])}$$

$$\left(\sum_n \sum_K \overline{\text{Res}}_{n,K} \|\boldsymbol{\varphi}^{\Delta x}\|_{H^1(K \times I^n)} \frac{\left(\int_K \int_{I^n} \|\nabla_{xt} \mathbf{V}\|^2 dx dt \right)^{\frac{1}{2}}}{\left(\int_K \int_{I^n} \|\nabla_{xt} \mathbf{V}\|^2 dx dt \right)^{\frac{1}{2}} + \Delta x^\theta} \right)$$

$$\leq C(\Delta x)^{\frac{1}{2}-\alpha} \left(\sum_{n,K} \|\boldsymbol{\varphi}^{\Delta x}\|^2_{H^1(K \times I^n)} \right)^{\frac{1}{2}} \left(\Delta x \sum_{n,K} \int_{I^n} \int_K \|\text{Res}\|^2 dx dt \right)^{\frac{1}{2}}$$

$$\leq C(\Delta x)^{\frac{1}{2}-\alpha} \|\boldsymbol{\varphi}\|_{H^1(\Omega \times [0,T])} \to 0 \quad \text{as } \Delta x \to 0.$$

Here, the last estimate follows the stability of the projection operator and from the weak BV estimate (3.3).

Similarly, define

$$
\mathcal{B}_{SC,2}(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x}) := \sum_n \sum_K \int_{I^n} \int_K D_{n,K}^{SC,2} \left(\langle \boldsymbol{\varphi}_t^{\Delta x}, \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_t^{\Delta x} \rangle + \sum_{k=1}^d \langle \boldsymbol{\varphi}_{x_k}^{\Delta x}, \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_{x_k}^{\Delta x} \rangle \right) dx dt,
$$

with

$$
D_{n,K}^{SC,2} = \frac{(\Delta x)^{\frac{1}{2}-\alpha} C^{SC} \overline{\mathrm{BRes}}_{n,K}}{\sqrt{\int_{I^n} \int_K \left(\langle \mathbf{V}_t^{\Delta x}, \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_t^{\Delta x} \rangle + \sum_{k=1}^d \langle \mathbf{V}_{x_k}^{\Delta x}, \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_{x_k}^{\Delta x} \rangle \right) dx dt + \Delta x^\theta}},
$$

Using the strict positivity of $\mathbf{U_V}$, we obtain that,

$$
D_{n,K}^{SC,2} \le C \frac{(\Delta x)^{\frac{1}{2}-\alpha} C^{SC} \overline{\mathrm{BRes}}_{n,K}}{\left(\int_K \int_{I^n} \|\nabla_{xt} \mathbf{V}\|^2 dx dt \right)^{\frac{1}{2}} + \Delta x^\theta}.
$$

Therefore,

$$
\begin{aligned}
|\mathcal{B}_{SC,2}(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x})| &\le C(\Delta x)^{\frac{1}{2}-\alpha} \|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega \times [0,T])} \\
&\quad \left(\sum_n \sum_K \overline{\mathrm{BRes}}_{n,K} \|\boldsymbol{\varphi}^{\Delta x}\|_{H^1(K \times I^n)} \frac{\left(\int_K \int_{I^n} \|\nabla_{xt} \mathbf{V}\|^2 dx dt \right)^{\frac{1}{2}}}{\left(\int_K \int_{I^n} \|\nabla_{xt} \mathbf{V}\|^2 dx dt \right)^{\frac{1}{2}} + \Delta x^\theta} \right) \\
&\le C(\Delta x)^{\frac{1}{2}-\alpha} \left(\sum_{n,K} \|\boldsymbol{\varphi}^{\Delta x}\|_{H^1(K \times I^n)}^2 \right)^{\frac{1}{2}} \left(\sum_{n,K} |\overline{\mathrm{BRes}}_{n,K}|^2 \right)^{\frac{1}{2}} \\
&\le C(\Delta x)^{\frac{1}{2}-\alpha} \|\boldsymbol{\varphi}\|_{H^1(\Omega \times [0,T])} \to 0 \quad \text{as } \Delta x \to 0.
\end{aligned}
$$

Here, the last estimate follows the stability of the projection operator and from the fact that the estimate on the $\overline{\mathrm{BRes}}$ term is a straightforward consequence of the weak BV estimate (3.3).

Claim 3 Let a part of the DG form (2.4) be defined as,

$$
\mathcal{B}_{DG}^{s1}(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x}) = \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left(\sum_{k=1}^d \langle \mathbb{F}^{k,*}(\mathbf{V}_{K,-}^{\Delta x}, \mathbf{V}_{K,+}^{\Delta x}), \boldsymbol{\varphi}_{K,-}^{\Delta x} \rangle \nu_{KK'}^k \right) d\sigma(x) dt \tag{4.8}
$$

Then,

$$
\mathcal{B}_{DG}^{s1}\left(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x} \right) \to 0 \quad \text{as } \Delta x \to 0. \tag{4.9}
$$

Switching the normal direction, we perform the following calculation,

$$
\begin{aligned}
\mathcal{B}_{DG}^{s1}(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x}) &= \frac{1}{2} \sum_{n} \sum_{K} \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left(\sum_{k=1}^{d} \langle \mathbb{F}^{k,*}(\mathbf{V}_{K,-}^{\Delta x}, \mathbf{V}_{K,+}^{\Delta x}), \boldsymbol{\varphi}_{K,-}^{\Delta x} \rangle v_{KK'}^k \right) d\sigma(x) dt \\
&\quad - \frac{1}{2} \sum_{n} \sum_{K} \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left(\sum_{k=1}^{d} \langle \mathbb{F}^{k,*}(\mathbf{V}_{K,-}^{\Delta x}, \mathbf{V}_{K,+}^{\Delta x}), \boldsymbol{\varphi}_{K,+}^{\Delta x} \rangle v_{KK'}^k \right) d\sigma(x) dt
\end{aligned}
$$

Adding and subtracting $\boldsymbol{\varphi}$ (using the continuity of the smooth test function $\boldsymbol{\varphi}$ across every edge) to the above, results in

$$
\begin{aligned}
|\mathcal{B}_{DG}^{s1}(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x})| &\leq \frac{1}{2} \left| \sum_{n} \sum_{K} \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left(\sum_{k=1}^{d} \langle \mathbb{F}^{k,*}(\mathbf{V}_{K,-}^{\Delta x}, \mathbf{V}_{K,+}^{\Delta x}), (\boldsymbol{\varphi}_{K,-}^{\Delta x} - \boldsymbol{\varphi}) \rangle v_{KK'}^k \right) d\sigma(x) dt \right| \\
&\quad + \frac{1}{2} \left| \sum_{n} \sum_{K} \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left(\sum_{k=1}^{d} \langle \mathbb{F}^{k,*}(\mathbf{V}_{K,-}^{\Delta x}, \mathbf{V}_{K,+}^{\Delta x}), (\boldsymbol{\varphi}_{K,+}^{\Delta x} - \boldsymbol{\varphi}) \rangle v_{KK'}^k \right) d\sigma(x) dt \right| \\
&\leq C \|\mathbf{V}^{\Delta x}\|_{L^{\infty}(\Omega \times [0,T])} \sum_{n} \sum_{K} (\text{meas}(\partial(I^n \times K)))^{\frac{1}{2}} \|\boldsymbol{\varphi}^{\Delta x} - \boldsymbol{\varphi}\|_{L^2(\partial(K \times I^n))}, \\
&\leq C \Delta x^{\frac{3}{2}} \|\mathbf{V}^{\Delta x}\|_{L^{\infty}(\Omega \times [0,T])} \sum_{n} \sum_{K} (\text{meas}(\partial(I^n \times K)))^{\frac{1}{2}} \|\boldsymbol{\varphi}\|_{H^2(K \times I^n)}, \\
&\leq C \Delta x \|\mathbf{V}^{\Delta x}\|_{L^{\infty}(\Omega \times [0,T])} \|\boldsymbol{\varphi}\|_{H^2(\Omega \times [0,T])} \to 0 \text{ as } \Delta x \to 0,
\end{aligned}
$$

Note that in the above estimate, we have used a stronger version of the approximation property (4.5):

$$
\|\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\Delta x}\|_{L^2(\partial(K \times I^n))} \leq C \Delta x^{\frac{3}{2}} \|\boldsymbol{\varphi}\|_{H^2(K \times I^n)}, \tag{4.10}
$$

which follows from the regularity of solutions of the elliptic equation (4.5a) that defines the projection operator and from the fact that the test function $\boldsymbol{\varphi}$ is infinitely differentiable.

Claim 4 Defining a part of the DG form as

$$
\mathcal{B}_{DG}^{s2}(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x}) = -\frac{1}{2} \sum_{n} \sum_{K} \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \langle \boldsymbol{\varphi}_{K,-}^{\Delta x}, \mathbf{D} \left(\mathbf{V}_{K,+}^{\Delta x} - \mathbf{V}_{K,-}^{\Delta x} \right) \rangle d\sigma(x) dt, \tag{4.11}
$$

we have

$$
\mathcal{B}_{DG}^{s2}(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x}) \to 0 \qquad \text{as } \Delta x \to 0. \tag{4.12}
$$

We proceed as in the previous claim to calculate,

$$
|\mathcal{B}_{DG}^{s2}(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x})|
$$

$$
\leq C\|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega\times[0,T])}\left|\sum_n\sum_K\sum_{K'\in\mathcal{N}(K)}\int_{I^n}\int_{\partial_{KK'}} \langle(\boldsymbol{\varphi}_{K,-}^{\Delta x}-\boldsymbol{\varphi}), (\mathbf{V}_{K,+}^{\Delta x}-\mathbf{V}_{K,-}^{\Delta x})\rangle d\sigma(x)dt\right|
$$

$$
+C\|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega\times[0,T])}\left|\sum_n\sum_K\sum_{K'\in\mathcal{N}(K)}\int_{I^n}\int_{\partial_{KK'}} \langle(\boldsymbol{\varphi}_{K,+}^{\Delta x}-\boldsymbol{\varphi}), (\mathbf{V}_{K,+}^{\Delta x}-\mathbf{V}_{K,-}^{\Delta x})\rangle d\sigma(x)dt\right|
$$

$$
\leq C\left(\sum_n\sum_K\sum_{K'\in\mathcal{N}(K)}\int_{I^n}\int_{\partial_{KK'}} \|\mathbf{V}_{K,+}^{\Delta x}-\mathbf{V}_{K,-}^{\Delta x}\|^2 d\sigma(x)dt\right)^{\frac{1}{2}}
$$

$$
\left(\sum_n\sum_K\sum_{K'\in\mathcal{N}(K)} \|\boldsymbol{\varphi}_{K,+}^{\Delta x}-\boldsymbol{\varphi}\|^2_{L^2(\partial_{KK'}\times I^n)}\right)^{\frac{1}{2}}
$$

$$
\leq C\Delta x^{\frac{1}{2}}\left(\sum_n\sum_K\sum_{K'\in\mathcal{N}(K)}\int_{I^n}\int_{\partial_{KK'}} \|\mathbf{V}_{K,+}^{\Delta x}-\mathbf{V}_{K,-}^{\Delta x}\|^2 d\sigma(x)dt\right)^{\frac{1}{2}}
$$

$$
\|\boldsymbol{\varphi}\|_{H^1(\Omega\times[0,T])} \quad \text{(by(4.5))}
$$

$$
\to 0, \quad \text{as } \Delta x \to 0, \quad \text{(by the weak BV estimate (3.3)).}
$$

Claim 5 Defining a part of the DG form (2.4) as

$$
\mathcal{B}_{DG}^{t1}(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x}) = \sum_n\sum_K\int_K \langle\mathbf{U}_{n+1,-}^{\Delta x}, \boldsymbol{\varphi}_{n+1,-}^{\Delta x}\rangle dx - \sum_n\sum_K\int_K \langle\mathbf{U}_{n,-}^{\Delta x}, \boldsymbol{\varphi}_{n,+}^{\Delta x}\rangle dx
\tag{4.13}
$$

we have

$$
\mathcal{B}_{DG}^{t1}(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x}) \to 0 \quad \text{as } \Delta x \to 0.
\tag{4.14}
$$

Adding and subtracting $\boldsymbol{\varphi}_n = \boldsymbol{\varphi}(t^n)$ and using the fact that $\boldsymbol{\varphi}$ is compactly supported in $(0, T)$,

$$
|\mathcal{B}_{DG}^{t1}(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x})| \leq \sum_n\sum_K\int_K |\langle\mathbf{U}_{n,-}^{\Delta x}, \boldsymbol{\varphi}_{n,+}^{\Delta x}-\boldsymbol{\varphi}_n\rangle|dx
$$

$$
+\sum_n\sum_K\int_K |\langle\mathbf{U}_{n+1,-}^{\Delta x}, \boldsymbol{\varphi}_{n+1,-}^{\Delta x}-\boldsymbol{\varphi}_{n+1}\rangle|dx
$$

$$
\leq C\|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega\times[0,T])}\sum_n
$$

$$
\times\sum_K(\text{meas}(\partial(I^n\times K)))^{\frac{1}{2}}(\|\boldsymbol{\varphi}_{n,-}^{\Delta x}-\boldsymbol{\varphi}_n\|_{L^2(K)}+\|\boldsymbol{\varphi}_{n+1,+}^{\Delta x}-\boldsymbol{\varphi}_{n+1}\|_{L^2(K)})
$$

$$\leq C \Delta x \|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega \times [0,T])} \|\boldsymbol{\varphi}\|_{H^2(\Omega \times (0,T))} \quad (\text{by}(4.10)),$$
$$\to 0 \quad \text{as } \Delta x \to 0,$$

Claim 6 The following bilinear form,

$$\overline{\mathcal{B}}_{DG}(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi} - \boldsymbol{\varphi}^{\Delta x}) \; = -\sum_n \sum_K \int_{I^n} \int_K (\langle \mathbf{U}(\mathbf{V}^{\Delta x}), (\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\Delta x})_t \rangle$$
$$+ \sum_{k=1}^d \langle \mathbf{F}(V^{\Delta x}), (\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\Delta x})_{x_k} \rangle) dx dt \qquad (4.15)$$

satisfies

$$\overline{\mathcal{B}}_{DG} \left(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi} - \boldsymbol{\varphi}^{\Delta x} \right) \to 0 \quad \text{as } \Delta x \to 0. \qquad (4.16)$$

We estimate,

$$|\overline{\mathcal{B}}_{DG} \left(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi} - \boldsymbol{\varphi}^{\Delta x} \right)| \leq \sum_n \sum_K \int_{I^n} \int_K \left| \left(\langle \mathbf{U}(\mathbf{V}^{\Delta x}), (\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\Delta x})_t \rangle \right. \right.$$
$$+ \sum_{k=1}^d \left\langle \mathbf{F}^k(\mathbf{V}^{\Delta x}), (\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\Delta x})_{x_k} \right\rangle \right) \bigg| dx dt$$
$$\leq C \|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega \times [0,T])} \|\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\Delta x}\|_{H^1(\Omega \times [0,T])}$$
$$\leq C \Delta x \|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega \times [0,T])} \|\boldsymbol{\varphi}\|_{H^2(\Omega \times (0,T))} \to 0 \qquad \text{as } \Delta x \to 0.$$

Here, we have used the stronger approximation result:

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\Delta x}\|_{H^1(K \times I^n)} \leq C \Delta x \|\boldsymbol{\varphi}\|_{H^2(K \times I^n)}, \qquad (4.17)$$

which follows from the regularity of the solutions to the elliptic equation (4.5a) and the fact that the test function $\boldsymbol{\varphi}$ is infinitely differentiable.

To prove convergence, we observe that

$$\int_0^T \int_\Omega \left(\langle \mathbf{U}(\mathbf{V}^{\Delta x}), \boldsymbol{\varphi}_t \rangle + \sum_k \langle \mathbf{F}^k(\mathbf{V}^{\Delta x}), \boldsymbol{\varphi}_{x_k} \rangle \right) dx dt$$
$$= \sum_n \sum_K \int_{I^n} \int_K \left(\langle \mathbf{U}(\mathbf{V}^{\Delta x}), \boldsymbol{\varphi}_t \rangle + \sum_k \langle \mathbf{F}^k(\mathbf{V}^{\Delta x}), \boldsymbol{\varphi}_{x_k} \rangle \right) dx dt$$
$$= -\overline{\mathcal{B}}_{DG} \left(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi} - \boldsymbol{\varphi}^{\Delta x} \right) - \mathcal{B} \left(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x} \right)$$
$$+ \mathcal{B}_{DG}^{t1} \left(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x} \right) + \mathcal{B}_{DG}^{s1} \left(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x} \right) + \mathcal{B}_{DG}^{s2} \left(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x} \right)$$

$$+\mathcal{B}_{SD}\left(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x}\right) + \mathcal{B}_{SC}\left(\mathbf{V}^{\Delta x}, \boldsymbol{\varphi}^{\Delta x}\right)$$
$$\to 0 \quad \text{as } \Delta x \to 0, \quad \text{(by (2.3; 4.6; 4.7; 4.9; 4.12; 4.14; 4.16))}.$$

Hence

$$\int_0^T \int_\Omega \left(\langle \mathbf{U}(\mathbf{V}^{\Delta x}), \boldsymbol{\varphi}_t \rangle + \sum_k \langle \mathbf{F}^k(\mathbf{V}^{\Delta x}), \boldsymbol{\varphi}_{x_k} \rangle\right) dx dt \to 0 \quad \text{as } \Delta x \to 0.$$

On account of the uniform L^∞ bound, we have that there exists a young measure μ such that $\mathbf{V}^{\Delta x} \rightharpoonup \mu$, in the sense of measures, as $\Delta x \to 0$. Furthermore as the nonlinearities commute with convergence in the sense of young measures [7], we have that

$$\lim_{\Delta x \to 0} \int_0^T \int_\Omega \left(\langle \mathbf{U}(\mathbf{V}^{\Delta x}), \boldsymbol{\varphi}_t \rangle + \sum_k \langle \mathbf{F}^k(\mathbf{V}^{\Delta x}), \boldsymbol{\varphi}_{x_k} \rangle\right) dx dt$$
$$= \int_0^T \int_\Omega \left(\langle \langle \mathbf{U}, \boldsymbol{\mu}_{x,t} \rangle, \boldsymbol{\varphi}_t \rangle + \sum_{k=1}^d \langle \langle \mathbf{F}^k, \boldsymbol{\mu}_{x,t} \rangle, \boldsymbol{\varphi}_{x_k} \rangle\right) dx dt$$

Hence, combining the above two limits, we see that the young measure μ is a measure valued solution (1.6) of (1.1). □

The next step in the analysis is to show consistency with the entropy condition (1.7). We do so in the following theorem.

Theorem 4.2 *Let $\mathbf{V}^{\Delta x}$ be the approximate solutions generated by the scheme (2.3). We assume that the parameter $\alpha > 0$ in (2.11b) and the approximate solutions are uniformly bounded (4.3). Then, the limit measure valued solution μ satisfies the entropy condition (1.7).*

Proof We consider a smooth test function $\varphi \geq 0$ and compactly supported in $\Omega \times (0, T)$ and show consistency with the entropy condition (1.7) in the following series of claims.

Claim 1 Let the DG quasilinear form be defined as in (2.4), then

$$\lim_{\Delta x \to 0} \mathcal{B}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}\varphi) \geq -\int_0^T \int_\Omega \left(\langle S, \boldsymbol{\mu}_{x,t} \rangle \varphi_t + \sum_{k=1}^d \langle Q^k, \boldsymbol{\mu}_{x,t} \rangle \varphi_{x_k}\right) dx dt \quad (4.18)$$

To prove the claim we define $\varphi^n = \varphi(t^n)$ and start with the following computation,

$$\mathcal{B}_{DG}^t(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}\varphi) = -\sum_n \sum_K \int_{I^n} \int_K \langle \mathbf{U}(\mathbf{V}^{\Delta x}), (\mathbf{V}^{\Delta x}\varphi)_t \rangle dx dt$$
$$+ \sum_n \sum_K \int_K \langle \mathbf{U}(\mathbf{V}_{n+1,-}^{\Delta x}), \mathbf{V}_{n+1,-}^{\Delta x}\varphi^{n+1} \rangle dx$$

$$-\sum_n \sum_K \int_K \left\langle \mathbf{U}(\mathbf{V}_{n,-}^{\Delta x}), \mathbf{V}_{n,+}^{\Delta x}\varphi^n \right\rangle dx$$

$$= \sum_n \sum_K \int_{I^n} \int_K \left\langle \mathbf{U}_t(\mathbf{V}^{\Delta x}), \mathbf{V}^{\Delta x}\varphi \right\rangle dx dt \quad \text{(integrating by parts)}$$

$$-\sum_n \sum_K \int_K \left\langle \mathbf{U}(\mathbf{V}_{n+1,-}^{\Delta x}), \mathbf{V}_{n+1,-}^{\Delta x}\varphi^{n+1} \right\rangle dx$$

$$+\sum_n \sum_K \int_K \left\langle \mathbf{U}(\mathbf{V}_{n,+}^{\Delta x}), \mathbf{V}_{n,+}^{\Delta x}\varphi^n \right\rangle dx$$

$$+\sum_n \sum_K \int_K \left\langle \mathbf{U}(\mathbf{V}_{n+1,-}^{\Delta x}), \mathbf{V}_{n+1,-}^{\Delta x}\varphi^{n+1} \right\rangle dx$$

$$-\sum_n \sum_K \int_K \left\langle \mathbf{U}(\mathbf{V}_{n,-}^{\Delta x}), \mathbf{V}_{n,+}^{\Delta x}\varphi^n \right\rangle dx$$

$$= \sum_n \sum_K \int_{I^n} \int_K S(\mathbf{U}(\mathbf{V}^{\Delta x}))_t \varphi dx dt \quad \text{(Definition of entropy function)}$$

$$-\sum_n \sum_K \int_K \left\langle (\mathbf{U}(\mathbf{V}_{n,-}^{\Delta x}) - \mathbf{U}(\mathbf{V}_{n,+}^{\Delta x})), \mathbf{V}_{n,+}^{\Delta x}\varphi^n \right\rangle dx$$

$$= -\sum_n \sum_K \int_{I^n} \int_K S(\mathbf{V}^{\Delta x})\varphi_t dx dt \quad \text{(Integrating by parts)}$$

$$+\sum_n \sum_K \int_K S(\mathbf{V}_{n+1,-}^{\Delta x})\varphi^{n+1} dx - \sum_n \sum_K \int_K S(\mathbf{V}_{n,+}^{\Delta x})\varphi^n dx$$

$$-\sum_n \sum_K \int_K \left\langle (\mathbf{U}(\mathbf{V}_{n,-}^{\Delta x}) - \mathbf{U}(\mathbf{V}_{n,+}^{\Delta x})), \mathbf{V}_{n,+}^{\Delta x}\varphi^n \right\rangle dx$$

$$= -\sum_n \sum_K \int_{I^n} \int_K S(\mathbf{V}^{\Delta x})\varphi_t dx dt$$

$$+\sum_n \sum_K \int_K (S(\mathbf{U}(\mathbf{V}_{n,-}^{\Delta x})) - S(\mathbf{U}(\mathbf{V}_{n,+}^{\Delta x})))$$

$$-\langle (\mathbf{U}(\mathbf{V}_{n,-}^{\Delta x}) - \mathbf{U}(\mathbf{V}_{n,+}^{\Delta x})), \mathbf{V}_{n,+}^{\Delta x})\rangle \varphi^n dx$$

$$= -\sum_n \sum_K \int_{I^n} \int_K S(\mathbf{V}^{\Delta x})\varphi_t dx dt$$

$$+\sum_n \sum_K \int_K \left(\int_0^1 \left\langle \left(\mathbf{V}_{n,-}^{\Delta x} - \mathbf{V}_{n,+}^{\Delta x}\right), \mathbf{U}_\mathbf{V}(\theta)\left(\mathbf{V}_{n,-}^{\Delta x} - \mathbf{V}_{n,+}^{\Delta x}\right) \right\rangle d\theta \right) \varphi^n dx$$

$$\geq -\int_0^T \int_\Omega S(\mathbf{V}^{\Delta x})\varphi_t dx dt \tag{4.19}$$

The last step follows from the strict positivity of $\mathbf{U_V}$ and positivity of the test function φ.

As

$$\sum_n \sum_K \int_{I^n} \int_K \sum_{k=1}^d \langle \mathbf{F}^k(\mathbf{V}^{\Delta x}), (\mathbf{V}^{\Delta x}\varphi)_{x_k} \rangle dxdt$$

$$= -\sum_n \sum_K \int_{I^n} \int_K \sum_{k=1}^d \langle (\mathbf{F}^k(\mathbf{V}^{\Delta x}))_{x_k}, \mathbf{V}^{\Delta x}\varphi \rangle dxdt$$

$$+\sum_n \sum_K \sum_{K'\in\mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left(\sum_{k=1}^d \langle \mathbf{F}^k(\mathbf{V}^{\Delta x}_{K,-}), \mathbf{V}^{\Delta x}_{K,-} \rangle v^k_{KK'} \right) \varphi d\sigma(x) dt$$

$$= -\sum_n \sum_K \int_{I^n} \int_K \sum_{k=1}^d Q^k(\mathbf{V}^{\Delta x})_{x_k} \varphi dxdt \quad \text{(definition of entropy flux)}$$

$$+\sum_n \sum_K \sum_{K'\in\mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left(\sum_{k=1}^d \langle \mathbf{F}^k(\mathbf{V}^{\Delta x}_{K,-}), \mathbf{V}^{\Delta x}_{K,-} \rangle \varphi v^k_{KK'} \right) d\sigma(x) dt$$

$$= \sum_n \sum_K \int_{I^n} \int_K \sum_{k=1}^d Q^k(\mathbf{V}^{\Delta x}) \varphi_{x_k} dxdt$$

$$+\sum_n \sum_K \sum_{K'\in\mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \sum_{k=1}^d \left(\langle \mathbf{F}^k(\mathbf{V}^{\Delta x}_{K,-}), \mathbf{V}^{\Delta x}_{K,-} \rangle - Q^k(\mathbf{V}^{\Delta x}_{K,-}) \right) \varphi v^k_{KK'} d\sigma(x) dt$$

$$= \sum_n \sum_K \int_{I^n} \int_K \sum_{k=1}^d Q^k(\mathbf{V}^{\Delta x}) \varphi_{x_k} dxdt$$

$$+\sum_n \sum_K \sum_{K'\in\mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \sum_{k=1}^d \Psi^k(\mathbf{V}^{\Delta x}_{K,-}) \varphi v^k_{KK'} d\sigma(x) dt. \quad (4.20)$$

Repeating the calculation in (3.11) and (3.12), we obtain,

$$\mathcal{B}^s_{DG}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}\varphi) = -\sum_n \sum_K \int_{I^n} \int_K \sum_{k=1}^d \langle \mathbf{F}^k(\mathbf{V}^{\Delta x}), (\mathbf{V}^{\Delta x}\varphi)_{x_k} \rangle dxdt$$

$$+\sum_n \sum_K \sum_{K'\in\mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left(\sum_{k=1}^d \langle \mathbb{F}^{k,*}(\mathbf{V}^{\Delta x}_{K,-}, \mathbf{V}^{\Delta x}_{K,+}), \mathbf{V}^{\Delta x}_{K,-}\varphi \rangle v^k_{KK'} \right) d\sigma(x) dt$$

$$-\frac{1}{2}\sum_n \sum_K \sum_{K'\in\mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \langle \mathbf{V}^{\Delta x}_{K,-}, \mathbf{D}(\mathbf{V}^{\Delta x}_{K,+} - \mathbf{V}^{\Delta x}_{K,-}) \rangle d\sigma(x) dt$$

$$= -\sum_n \sum_K \int_{I^n} \int_K \sum_{k=1}^d Q^k(\mathbf{V}^{\Delta x})\varphi_{x_k}\,dx\,dt$$

$$+\sum_n \sum_K \sum_{K'\in\mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left(\sum_{k=1}^d (\langle \mathbb{F}^{k,*}(\mathbf{V}^{\Delta x}_{K,-},\mathbf{V}^{\Delta x}_{K,+}),\mathbf{V}^{\Delta x}_{K,-}\rangle \right.$$

$$\left. -\,\Psi^k(\mathbf{V}^{\Delta x}_{K,-}))v^k_{KK'}\right)\varphi\,d\sigma(x)\,dt$$

$$-\frac{1}{2}\sum_n \sum_K \sum_{K'\in\mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left\langle \mathbf{V}^{\Delta x}_{K,-}, \mathbf{D}(\mathbf{V}^{\Delta x}_{K,+}-\mathbf{V}^{\Delta x}_{K,-})\right\rangle d\sigma(x)\,dt$$

$$\geq -\sum_n \sum_K \int_{I^n} \int_K \sum_{k=1}^d Q^k(\mathbf{V}^{\Delta x})\varphi_{x_k}\,dx\,dt \tag{4.21}$$

Using the definition of the residual (2.9) and the streamline diffusion operator, we have

$$\mathcal{B}_{SD}(\mathbf{V}^{\Delta x},\mathbf{V}^{\Delta x}\varphi) = \sum_n \sum_K \int_{I^n} \int_K \left\langle \left(\mathbf{U}(\mathbf{V}^{\Delta x})(\mathbf{V}^{\Delta x}\varphi)_t + \sum_{k=1}^d \mathbf{F}^k_{\mathbf{V}}(\mathbf{V}^{\Delta x})(\mathbf{V}^{\Delta x}\varphi)_{x_k}\right),\mathbf{D}^{SD}\mathrm{Res}\right\rangle dx\,dt$$

$$= \sum_n \sum_K \int_{I^n} \int_K \langle \mathrm{Res},\mathbf{D}^{SD}\mathrm{Res}\rangle\varphi\,dx\,dt$$

$$+\underbrace{\sum_n \sum_K \int_{I^n} \int_K \left\langle \left(\mathbf{U}(\mathbf{V}^{\Delta x})\mathbf{V}^{\Delta x}\varphi_t + \sum_{k=1}^d \mathbf{F}^k_{\mathbf{V}}(\mathbf{V}^{\Delta x})\mathbf{V}^{\Delta x}\varphi_{x_k}\right),\mathbf{D}^{SD}\mathrm{Res}\right\rangle dx\,dt}_{T_1} \geq T_1. \tag{4.22}$$

However repeating the calculations in the proof of (4.6), we see that

$$|T_1| \leq C\|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega\times[0,T])}\Delta x^{\frac{1}{2}}\left(\sum_{n,K}\|\boldsymbol{\varphi}^{\Delta x}\|^2_{H^1(K\times I^n)}\right)^{\frac{1}{2}}\left(\Delta x\sum_{n,K}\int_{I^n}\int_K \|\mathrm{Res}\|^2 dx\,dt\right)^{\frac{1}{2}} \tag{4.23}$$

$$\leq C\|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega\times[0,T])}\Delta x^{\frac{1}{2}}\|\boldsymbol{\varphi}\|_{H^1(\Omega\times[0,T])} \to 0 \quad \text{as } \Delta x \to 0.$$

Next, from the definition of the shock capturing operator and strict positivity of $\mathbf{U}_{\mathbf{V}}$, we obtain

$$\mathcal{B}_{SC}(\mathbf{V}^{\Delta x},\mathbf{V}^{\Delta x}\varphi) = \sum_n \sum_K \int_{I^n} \int_K D^{SC}_{n,K}(\langle(\mathbf{V}^{\Delta x}\varphi)_t,\mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,K})\mathbf{V}^{\Delta x}_t\rangle + \sum_{k=1}^d \langle(\mathbf{V}^{\Delta x}\varphi)_{x_k},\mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,K})\mathbf{V}^{\Delta x}_{x_k}\rangle)\,dx\,dt$$

$$= \sum_n \sum_K \int_{I^n} \int_K D^{SC}_{n,K}(\langle\mathbf{V}^{\Delta x}_t,\mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,K})\mathbf{V}^{\Delta x}_t\rangle + \sum_{k=1}^d \langle\mathbf{V}^{\Delta x}_{x_k},\mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,K})\mathbf{V}^{\Delta x}_{x_k}\rangle)\varphi\,dx\,dt$$

$$+\underbrace{\sum_n \sum_K \int_{I^n} \int_K D^{SC}_{n,K}(\langle\mathbf{V}^{\Delta x}\varphi_t,\mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,K})\mathbf{V}^{\Delta x}_t\rangle + \sum_{k=1}^d \langle\mathbf{V}^{\Delta x}\varphi_{x_k},\mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,K})\mathbf{V}^{\Delta x}_{x_k}\rangle)\,dx\,dt}_{T_2} \geq T_2$$

$$\tag{4.24}$$

Repeating the calculation in the proof of (4.7), we obtain

$$
\begin{aligned}
|T_2| \;&\le\; C(\Delta x)^{\frac{1}{2}-\alpha}\|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega\times[0,T])}\|\varphi\|_{H^1(\Omega\times[0,T])}\\
&\quad\left(\left(\Delta x\sum_{n,K}\int_{I^n}\int_K\|\mathrm{Res}\|^2 dxdt\right)^{\frac{1}{2}}+\left(\sum_{n,K}|\overline{\mathrm{BRes}}_{n,K}|^2\right)^{\frac{1}{2}}\right)\\
&\le\; C(\Delta x)^{\frac{1}{2}-\alpha}\|\varphi\|_{H^1(\Omega\times[0,T])}\to 0\quad\text{as }\Delta x\to 0.
\end{aligned}\tag{4.25}
$$

From (4.19), (4.21), (4.22) and (4.24), we have

$$
\begin{aligned}
\mathcal{B}(\mathbf{V}^{\Delta x},\mathbf{V}^{\Delta x}\varphi) &= \mathcal{B}^t_{DG}(\mathbf{V}^{\Delta x},\mathbf{V}^{\Delta x}\varphi)+\mathcal{B}^s_{DG}(\mathbf{V}^{\Delta x},\mathbf{V}^{\Delta x}\varphi)\\
&\quad +\mathcal{B}_{SD}(\mathbf{V}^{\Delta x},\mathbf{V}^{\Delta x}\varphi)+\mathcal{B}_{SC}(\mathbf{V}^{\Delta x},\mathbf{V}^{\Delta x}\varphi)\\
&\ge -\left(\int_0^T\int_\Omega\left(S(\mathbf{V}^{\Delta x})\varphi_t+\sum_{k=1}^d Q^k(\mathbf{V}^{\Delta x})\varphi_{x_k}\right)dxdt\right)+T_1+T_2.
\end{aligned}
$$

Passing to the limit in the above expression as $\Delta x\to 0$, using the commutation of nonlinearities and limit in the sense of measures and using (4.23) and (4.25) results in,

$$
\lim_{\Delta x\to 0}\mathcal{B}\left(\mathbf{V}^{\Delta x},\mathbf{V}^{\Delta x}\varphi\right)\ge -\int_0^T\int_\Omega\left(\langle S,\boldsymbol{\mu}_{x,t}\rangle\varphi_t+\sum_{k=1}^d\langle Q^k,\boldsymbol{\mu}_{x,t}\rangle\varphi_{x_k}\right)dxdt,
$$

thus proving the claim.

Next, we use the H^1 projection operator $\Pi^{\Delta x}$ from (4.4) and show the following claim,

Claim 2 We have

$$
\lim_{\Delta x\to 0}\mathcal{B}(\mathbf{V}^{\Delta x},\mathbf{V}^{\Delta x}\varphi-\Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))=0.\tag{4.26}
$$

We start with calculations involving the DG quasilinear form (2.4). Let,

$$
\begin{aligned}
&\mathcal{B}^{int}_{DG}(\mathbf{V}^{\Delta x},(\mathbf{V}^{\Delta x}\varphi-\Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi)))\\
&= -\sum_n\sum_K\int_{I^n}\int_K(\langle\mathbf{U}(\mathbf{V}^{\Delta x}),(\mathbf{V}^{\Delta x}\varphi-\Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))_t\rangle+\sum_{k=1}^d\langle\mathbf{F}^k(\mathbf{V}^{\Delta x}),(\mathbf{V}^{\Delta x}\varphi-\Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))_{x_k}\rangle)dxdt\\
&= \underbrace{\sum_n\sum_K\int_{I^n}\int_K(\langle\mathbf{U}(\mathbf{V}^{\Delta x})_t,(\mathbf{V}^{\Delta x}\varphi-\Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))\rangle+\sum_{k=1}^d\langle\mathbf{F}^k(\mathbf{V}^{\Delta x})_{x_k},(\mathbf{V}^{\Delta x}\varphi-\Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))\rangle)dxdt}_{T_3}\\
&\quad -\sum_n\sum_K\int_K\langle\mathbf{U}(\mathbf{V}^{\Delta x}_{n+1,-}),(\mathbf{V}^{\Delta x}\varphi-\Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))^{n+1,-}\rangle dx
\end{aligned}
$$

$$+ \sum_n \sum_K \int_K \langle \mathbf{U}(\mathbf{V}_{n+}^{\Delta x}), (\mathbf{V}^{\Delta x}\varphi - \Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))^{n,+}\rangle dx$$

$$- \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left(\sum_{k=1}^d \langle \mathbf{F}^k(\mathbf{V}_{K,-}^{\Delta x}), (\mathbf{V}^{\Delta x}\varphi - \Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))_{K,-}\rangle v_{KK'}^k \right) d\sigma(x) dt \quad (4.27)$$

Using the definition of the residual (2.9), the weak BV estimate (3.3) and the projection error estimate (4.5b), we obtain,

$$|T_3| = \left| \sum_n \sum_K \int_{I^n} \int_K \left\langle \mathrm{Res}, (\mathbf{V}^{\Delta x}\varphi - \Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi)) \right\rangle dx dt \right|$$

$$\leq \sum_n \sum_K \overline{\mathrm{Res}}_{n,K} \| \mathbf{V}^{\Delta x}\varphi - \Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi) \|_{L^2(K \times I^n)}$$

$$\leq C\Delta x \sum_n \sum_K \overline{\mathrm{Res}}_{n,K} \| \nabla_{xt}(\mathbf{V}^{\Delta x}\varphi) \|_{L^2(K \times I^n)}$$

$$\leq C\Delta x^{\frac{1}{2}} \| \mathbf{V}^{\Delta x} \|_{L^\infty(\Omega \times [0,T])} \| \varphi \|_{H^1(\Omega \times (0,T))} \left(\Delta x \sum_{n,K} \int_{I^n} \int_K \| \mathrm{Res} \|^2 dx dt \right)^{\frac{1}{2}}$$

$$+ C\Delta x^{\alpha} \| \varphi \|_{L^\infty(\Omega \times [0,T]} \left(\Delta x^{1-\alpha} \sum_n \sum_K \overline{\mathrm{Res}}_{n,K} \left(\int_{I^n} \int_K \| \nabla_{xt}\mathbf{V}^{\Delta x} \|^2 dx dt \right)^{\frac{1}{2}} \right)$$

$$\to 0, \quad \text{as } \Delta x \to 0. \quad (4.28)$$

Using (4.27) and calculating with the full DG quasilinear form, we obtain

$$\mathcal{B}_{DG}(\mathbf{V}^{\Delta x}, (\mathbf{V}^{\Delta x}\varphi - \Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))) = T_3$$

$$- \sum_n \sum_K \int_K \left\langle \mathbf{U}(\mathbf{V}_{n+1,-}^{\Delta x}), (\mathbf{V}^{\Delta x}\varphi - \Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))^{n+1,-} \right\rangle dx$$

$$+ \sum_n \sum_K \int_K \left\langle \mathbf{U}(\mathbf{V}_{n+}^{\Delta x}), (\mathbf{V}^{\Delta x}\varphi - \Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))^{n,+} \right\rangle dx$$

$$- \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left(\sum_{k=1}^d \langle \mathbf{F}^k(\mathbf{V}_{K,-}^{\Delta x}), (\mathbf{V}^{\Delta x}\varphi - \Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))_{K,-}\rangle v_{KK'}^k \right) d\sigma(x) dt$$

$$+ \sum_n \sum_K \int_K \left\langle \mathbf{U}(\mathbf{V}_{n+1,-}^{\Delta x}), (\mathbf{V}^{\Delta x}\varphi - \Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))^{n+1,-} \right\rangle dx$$

$$- \sum_n \sum_K \int_K \left\langle \mathbf{U}(\mathbf{V}_{n-}^{\Delta x}), (\mathbf{V}^{\Delta x}\varphi - \Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))^{n,+} \right\rangle dx$$

$$+ \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \left(\sum_{k=1}^d \langle \mathbb{F}^{k,*}(\mathbf{V}_{K,-}^{\Delta x}, \mathbf{V}_{K,+}^{\Delta x}), (\mathbf{V}^{\Delta x}\varphi - \Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))_{K,-}\rangle \right.$$

$$\left. v_{KK'}^k \right) d\sigma(x) dt$$

$$-\frac{1}{2}\sum_n\sum_K\sum_{K'\in\mathcal{N}(K)}\int_{I^n}\int_{\partial_{KK'}}\big\langle(\mathbf{V}^{\Delta x}\varphi-\Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))_{K,-},\mathbf{D}\big(\mathbf{V}^{\Delta x}_{K,+}-\mathbf{V}^{\Delta x}_{K,-}\big)\big\rangle d\sigma(x)dt$$

$$=T_3+\sum_n\sum_K\int_K\big\langle\mathbf{U}(\mathbf{V}^{\Delta x}_{n,+})-\mathbf{U}(\mathbf{V}^{\Delta x}_{n,-}),(\mathbf{V}^{\Delta x}\varphi-\Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))^{n,+}\big\rangle dx$$

$$+\sum_n\sum_K\sum_{K'\in\mathcal{N}(K)}\int_{I^n}\int_{\partial_{KK'}}\Big(\sum_{k=1}^d\langle\mathbb{F}^{k,*}(\mathbf{V}^{\Delta x}_{K,-},\mathbf{V}^{\Delta x}_{K,+})-\mathbf{F}^k(\mathbf{V}^{\Delta x}_{K,-}),$$

$$(\mathbf{V}^{\Delta x}\varphi-\Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))_{K,-}\rangle v^k_{KK'}\Big)d\sigma(x)dt$$

$$-\frac{1}{2}\sum_n\sum_K\sum_{K'\in\mathcal{N}(K)}\int_{I^n}\int_{\partial_{KK'}}\big\langle(\mathbf{V}^{\Delta x}\varphi-\Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))_{K,-},\mathbf{D}(\mathbf{V}^{\Delta x}_{K,+}-\mathbf{V}^{\Delta x}_{K,-})\big\rangle d\sigma(x)dt$$

$$\tag{4.29}$$

Using the definition of the boundary residual in (2.11d), the weak BV estimate (3.3), the estimate (4.28) and the projection error estimate (4.5), the above can be simplified as

$$|\mathcal{B}_{DG}(\mathbf{V}^{\Delta x},(\mathbf{V}^{\Delta x}\varphi-\Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi)))|\le|T_3|$$
$$+\sum_n\sum_K\overline{\mathrm{BRes}}_{n,K}\|\mathbf{V}^{\Delta x}\varphi-\Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))\|_{L^2(\partial(I^n\times K))}$$
$$\le|T_3|+C\Delta x^{\frac{1}{2}}\sum_n\sum_K\overline{\mathrm{BRes}}_{n,K}\|\nabla_{xt}(\mathbf{V}^{\Delta x}\varphi)\|_{L^2(K\times I^n)}$$
$$\le|T_3|+C\Delta x^{\frac{1}{2}}\|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega\times[0,T])}\|\varphi\|_{H^1(\Omega\times(0,T))}\left(\sum_{n,K}\int_{I^n}\int_K|\overline{\mathrm{BRes}}_{n,K}|^2dxdt\right)^{\frac{1}{2}}$$
$$+C\Delta x^\alpha\|\varphi\|_{L^\infty(\Omega\times[0,T])}\left(\Delta x^{\frac{1}{2}-\alpha}\sum_n\sum_K\overline{\mathrm{BRes}}_{n,K}\left(\int_{I^n}\int_K\|\nabla_{xt}\mathbf{V}^{\Delta x}\|^2dxdt\right)^{\frac{1}{2}}\right)$$
$$\to0,\quad\text{as }\Delta x\to0.\tag{4.30}$$

Next, we consider the streamline diffusion quasilinear form (2.8) and use the stability of the projection operator (4.5) and the weak BV estimate (3.3) to obtain,

$$\big|\mathcal{B}_{SD}(\mathbf{V}^{\Delta x},(\mathbf{V}^{\Delta x}\varphi-\Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))\big|$$
$$=\left|\sum_n\sum_K\int_{I^n}\int_K\big\langle\Big(\mathbf{U}(\mathbf{V}^{\Delta x})(\mathbf{V}^{\Delta x}\varphi-\Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))_t\right.$$
$$+\sum_{k=1}^d\mathbf{F}^k_{\mathbf{V}}(\mathbf{V}^{\Delta x})(\mathbf{V}^{\Delta x}\varphi-\Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))_{x_k}\Big),\mathbf{D}^{SD}\mathrm{Res}\big\rangle dxdt\Bigg|$$
$$\le C\Delta x\|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega\times[0,T])}\sum_n\sum_K\|\nabla_{xt}(\mathbf{V}^{\Delta x}\varphi-\Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))\|_{L^2(K\times I^n)}\overline{\mathrm{Res}}_{n,K}$$
$$\le C\Delta x\|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega\times[0,T])}\sum_n\sum_K\|\nabla_{xt}\mathbf{V}^{\Delta x}\varphi\|_{L^2(K\times I^n)}\overline{\mathrm{Res}}_{n,K}$$

$$\leq C\Delta x^{\frac{1}{2}} \|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega \times [0,T])} \|\varphi\|_{H^1(\Omega \times (0,T))} \left(\Delta x \sum_{n,K} \int_{I^n} \int_K \|\mathrm{Res}\|^2 dx dt \right)^{\frac{1}{2}}$$

$$+ C\Delta x^\alpha \|\varphi\|_{L^\infty(\Omega \times [0,T]} \left(\Delta x^{1-\alpha} \sum_n \sum_K \overline{\mathrm{Res}}_{n,K} \left(\int_{I^n} \int_K \|\nabla_{xt} \mathbf{V}^{\Delta x}\|^2 dx dt \right)^{\frac{1}{2}} \right)$$

$$\to 0, \quad \text{as } \Delta x \to 0. \tag{4.31}$$

Given the orthogonality of the projection operator Π (4.5) and the definition of the shock capturing operator, it is easy to see that,

$$\mathcal{B}_{SC}(\mathbf{V}^{\Delta x}, (\mathbf{V}^{\Delta x}\varphi - \Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))) \equiv 0. \tag{4.32}$$

The claim (4.26) is proved by combining (4.30), (4.31) and (4.32).

Using (4.18), (4.26) and the definition of the scheme (2.3), we obtain,

$$\int_0^T \int_\Omega \left(\langle S, \mu_{x,t} \rangle \varphi_t + \sum_{k=1}^d \langle Q^k, \mu_{x,t} \rangle \varphi_{x_k} \right) dx dt \geq - \lim_{\Delta x \to 0} \mathcal{B}\left(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}\varphi\right)$$

$$\geq - \lim_{\Delta x \to 0} \mathcal{B}\left(\mathbf{V}^{\Delta x}, (\mathbf{V}^{\Delta x}\varphi - \Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi))\right)$$

$$- \lim_{\Delta x \to 0} \underbrace{\mathcal{B}\left(\mathbf{V}^{\Delta x}, \Pi^{\Delta x}(\mathbf{V}^{\Delta x}\varphi)\right)}_{=0}$$

$$\geq 0. \tag{4.33}$$

This proves (1.7). $\qquad \square$

Remark 4.3 The main assumption in the convergence to entropy measure valued solutions is the uniform L^∞ bound (4.3). This is assumption is true for scalar conservation laws and directly follows from the entropy bound (3.2) by choosing the entropy:

$$S(U) = -\log(b - U) - \log(U - a). \tag{4.34}$$

provided that the initial data $U_0(x)$ for the scalar conservation law $\mathbf{U} = U$ in (1.1) satisfy the bound,

$$a < U_0(x) < b, \quad \forall x \in \Omega,$$

for constants $a, b \in \mathbb{R}$. See [11] for a proof of how control on the above entropy implies an L^∞ bound on the approximate solutions.

However, for systems of conservation laws, it is still an open problem to prove L^∞ bounds, even at the level of the continuous problem. This bound does exist for some special systems [7] which possess invariant regions. The uniform L^∞ bound assumption is reasonable as we are interested in characterizing the consistency of the scheme (2.3).

Remark 4.4 Another assumption is the choice of the scaling $\alpha > 0$ in the shock capturing operator (2.11b). Note that this assumption is only required in controlling the projection error (4.26) in the proof of entropy consistency. Some numerical experiments have revealed that a non-zero but small value of α might result in a lower convergence rate, even for linear problems, see [16] for examples.

4.1 Convergence for scalar conservation laws

A slightly modified version of the shock capturing streamline diffusion DG scheme (2.3) converges to the entropy solutions of scalar conservation laws. The main modification lies in the choice of the entropy S with respect to which the entropy variables **V** and the scheme (2.3) is defined. As scalar conservation laws possess infinitely many entropies, we choose the standard quadratic entropy $S(U) = \frac{1}{2}U^2$ and the entropy variables $V = U$ in (2.3). Furthermore, the entropy conservative flux $\mathbb{F}^{k,*}$ in (2.4) is replaced by the arithmetic average:

$$\mathbb{F}^{k,*}(a, b) = \frac{F^k(a) + F^k(b)}{2}, \tag{4.35}$$

and the numerical diffusion operator in (2.7) is taken to be the numerical viscosity corresponding to the Godunov scheme [28]. With these modifications, we have the following convergence theorem in the scalar case,

Theorem 4.5 *Assume that the initial data $U_0(x)$ for the scalar conservation law* **U** $= U$ *in (1.1) satisfy the bound,*

$$a < U_0(x) < b, \quad \forall x \in \Omega,$$

for constants $a, b \in \mathbb{R}$. Let $U^{\Delta x}$ be the approximate solutions generated by the numerical scheme (2.3) with numerical flux (4.35) and numerical diffusion operator corresponding to the Godunov scheme, then the approximate solutions converge to the entropy solution of the underlying scalar conservation law, i.e, (1.1) with $m = 1$.

The proof of the entropy bound (3.2) for any entropy is a straightforward consequence of the proof of theorem 3.1. The argument of Tadmor [28] can be used of show that the scheme is consistent with any entropy as the numerical diffusion operator is the diffusion of the Godunov scheme. The L^∞ bound follows from the special choice of entropy (4.34). Following [3,7,19], convergence is a consequence of the following three steps,

1. *Convergence to measure valued solutions;* follows directly from Theorem 4.1.
2. *Consistency with entropy conditions:* straightforward modification of Theorem 4.2 to the scalar case, along with the choice of the Godunov flux leading to consistency with any convex entropy function.
3. *Consistency with initial data:* follows as in Sect. 5 of [19].

Given the above steps, the reduction theorem of [7] shows that the approximate solutions converge to the entropy solution of the underlying scalar conservation law.

Remark 4.6 The proof of convergence in the scalar case is essentially the same as that presented in [19]. The only difference lies in the choice of the shock capturing operator (2.11a). In [19], the authors require a lower bound on $D_{n,K}^{SC}$ in terms of $\Delta x^{\frac{3}{2}}$ (see Eq. (2.14) of [19]) which will reduce the convergence rate for smooth problems. We do not require such a lower bound.

4.2 Convergence for linear symmetrizable systems

Consider the linear symmetrizable system (4.1) and complete the scheme (2.3) with the entropy conservative flux:

$$\mathbb{F}^{k,*}(a, b) = \frac{1}{2}\left(A^k(a + b)\right),\tag{4.36}$$

Following [11], we see that the above flux satisfies (2.6) for the above entropy formulation (4.2). Observe that although the system (4.1) is linear, the streamline diffusion-shock capturing DG scheme (2.3) approximating it is nonlinear on account of the nonlinear shock capturing term (2.11a, 2.11b, 2.11c, 2.11d).

We have the following stability and convergence theorem for the scheme (2.3) with flux (4.36) approximating the linear symmetrizable system (4.1).

Theorem 4.7 *Consider the linear symmetrizable system* (4.1) *with symmetrizer B. Let* $\mathbf{U}^{\Delta x} = \mathbf{U}(\mathbf{V}^{\Delta x})$ *be the approximate solutions generated by the streamline diffusion-shock capturing DG scheme* (2.3) *with numerical flux* (2.6). *Then, the approximate solutions satisfy the following energy bounds,*

$$\|\mathbf{U}^{\Delta x}(., t_-^n)\|_{L^2(\Omega)} \le C\|\mathbf{U}^{\Delta x}(., t_-^0)\|_{L^2(\Omega)},\tag{4.37}$$

for all discrete time levels t^n.

Furthermore, the approximate solutions $\mathbf{U}^{\Delta x} \rightharpoonup \mathbf{U}$ *in* $L^2(\Omega \times [0, T])$ *and* $\mathbf{U}$ *is the unique weak solution of the system* (4.1).

The proof is just a repetition of Theorem 4.1, replacing the L^∞ bound with the L^2 bound at appropriate places and adding consistency with the initial data.

5 Details of implementation

5.1 The mesh and mesh generation

In one space dimensions we have (so far) only used regular meshes. In two spatial dimensions we have used the distmesh mesh generator [24], which returns a Delauney triangulation of the domain Ω based on a signed distance function.

As the DG method is fully implicit there is no CFL-like condition that needs to be satisfied. Nevertheless for accuracy reasons we use the following condition to determine the time step size:

$$\Delta t^{n+1} \leq C^{CFL} \min_{K \in \mathcal{T}, x \in K} \frac{\frac{|K|}{\Delta x_K}}{\lambda_{max}(\mathbf{U}^{\Delta x}(x, t^n))}, \tag{5.1}$$

where $\lambda_{max}(\mathbf{U}) = \max_v \lambda_{max}(\mathbf{U}; v)$ is the maximal wave speed in all directions and the constant C^{CFL} is typically chosen to be $1/2$.

5.2 Choice of basis functions

The approximate solution $\mathbf{V}^{\Delta x}$ is in $\mathcal{V}_p$. For the computation we express it as a linear combination of basis functions:

$$\mathbf{V}^{\Delta x} = \sum_{K,k,i,n} \hat{v}^n_{Kki} \boldsymbol{\phi}^n_{Kki}, \tag{5.2}$$

where $0 \leq n \leq N - 1$, $K \in \mathcal{T}$, $1 \leq k \leq m$ and $1 \leq i \leq n_f$ and the coefficients $\hat{v}^n_{Kki}$ are the degrees of freedom. The indices indicate the support of the basis functions: We choose the basis functions $\boldsymbol{\phi}^n_{Kki}$ such that they are nonzero only in one space-time element: $\boldsymbol{\phi}^n_{Kki}$ is nonzero in $K \times I^n$. Additionally, only one component is nonzero—namely the k-th component for $\boldsymbol{\phi}^n_{Kki}$—and we use the same scalar basis functions ϕ^n_{Ki} for all the components, i.e.

$$\left(\boldsymbol{\phi}^n_{Kki}\right)_l = \delta_{kl} \phi^n_{Ki}, \quad 1 \leq l \leq m. \tag{5.3}$$

The scalar basis functions have to span $\mathbb{P}_p(K \times I^n)$. We use monomials, except that they are shifted and scaled (for simplicity we consider here the two dimensional case):

$$\phi^n_{Ki}|_{K \times I^n} = \left(\frac{t - t^{n+1}}{\Delta t^n}\right)^{p^{t,i}} \left(\frac{x - \bar{x}_K}{\Delta x}\right)^{p^{x,i}} \left(\frac{y - \bar{y}_K}{\Delta x}\right)^{p^{y,i}} \tag{5.4}$$

where $(\bar{x}_K, \bar{y}_K)$ is the centroid of cell K and $p^{t,i}$, $p^{x,i}$, $p^{y,i}$ is the polynomial degree of the i-th scalar basis function in t, x, resp. y-direction. We use a maximal degree of p, hence:

$$p^{t,i} + p^{x,i} + p^{y,i} \leq p. \tag{5.5}$$

In general there are $n_f = \binom{1+d+p}{p}$ scalar basis functions. This leads to a total number of degrees of freedom of $N_c N m n_f = N_c N m \binom{1+d+p}{p}$, where $N_c = |\mathcal{T}|$ is the number of cells.

5.3 Quadrature rules

There are three types of integration that have to be performed, compare (2.4), (2.8), (2.11a, 2.11b, 2.11c, 2.11d): integration over the whole space-time element $K \times I^n$, integration over the spatial elements K, and integration over the edges of the elements

and the time $\partial_{KK'} \times I^n$. For the tensor product domains $K \times I^n$ and $\partial_{KK'} \times I^n$ we use tensor product integration formulas. Hence, we only need to further specify the quadrature rules for K, $\partial_{KK'}$ and I^n. For the integration over triangles K, Dunavant quadrature rules of order $2p + 2$ are applied [8], while for the integration over $\partial_{KK'}$ and I^n one dimensional Gaussian quadrature rules of order $2p + 2$ are used.

5.4 Nonlinear and linear solvers

Because the form $\mathcal{B}$ is linear in the test function, fulfilling the variational form (2.3) for all test functions $\mathbf{W}^{\Delta x} \in \mathcal{V}_p$ is equivalent to requiring that (2.3) is satisfied for all basis functions $\boldsymbol{\phi}^n_{Kki}$. Hence, we define $F^n_{K'lj} = \mathcal{B}(\mathbf{V}^{\Delta x}, \boldsymbol{\phi}^n_{K'lj})$. As we are able to march in time, due to (2.5), we consider each time step separately. In the n-th timestep we have to solve the nonlinear system for $\hat{v}^n$

$$F^n_{K'lj}(\hat{v}^n) = 0, \quad K' \in \mathcal{T}, 1 \le l \le m, 1 \le j \le n_f, \tag{5.6}$$

where $\hat{v}^n$ is the vector of the degrees of freedom for the n-th time slab.

We use a damped Newton method [15] to solve this system, the most important step is to solve for the Newton correction $\delta\hat{v}^n$ at the current state $\hat{v}^n$:

$$J^n(\hat{v}^n)\delta\hat{v}^n = F^n(\hat{v}^n), \tag{5.7}$$

where the Jacobian is given by

$$J^n_{K'lj,Kki} = (F_{K'lj})_{\hat{v}^n_{Kki}}, \quad K, K' \in \mathcal{T}, 1 \le k, l \le m, 1 \le i, j \le n_f \tag{5.8}$$

Currently the Jacobian is created analytically, which is quite complicated and time consuming, and a sparse LU factorization is used to solve the linearized system.

5.5 Choice of parameters

There are some parameters that need to be specified in order to complete the method (2.3). They have some influence on the results, but we have not tried to systematically study them and optimize them for each problem. Rather, we use problem independent values. The streamline diffusion form (2.8) involves the coefficient C^{SD}, while the coefficients C^{SC}, $\bar{C}^{SC}$ appear in the shock capturing term (2.11a, 2.11b, 2.11c, 2.11d). We set both coefficients C^{SD} and C^{SC} to 1, while $\bar{C}^{SC}$ is set to 0. The remaining parameters of the shock capturing operator are as follows: θ is $(d + 1)/2$, while α is set to 0.

6 Numerical experiments

We test the shock capturing streamline diffusion DG scheme (2.3) for systems of conservation laws, in both one and two space dimensions. We start with a linear system.

6.1 Wave equation

6.1.1 Exact form of equations, entropy formulation

The standard wave equation in two space dimensions can be written as,

$$h_t + cu_x + cv_y = 0 \tag{6.1a}$$

$$u_t + ch_x = 0 \tag{6.1b}$$

$$v_t + ch_y = 0 \tag{6.1c}$$

Here, h can be considered as a height or pressure variable and u, v are the velocities in the x- and y-directions respectively. Denoting the vector of conserved variables as $\mathbf{U}$, we see that the wave equation is already symmetric (i.e, the symmetrizer is the identity matrix) and the entropy is the energy: $S(\mathbf{U}) = \frac{1}{2}(h^2 + u^2 + v^2)$.

Therefore, the entropy conservative flux in (2.4) is the average flux of the two neighboring states. We use the following Rusanov type diffusion operator:

$$\mathbf{D}(a, b; \nu) = \max\{\lambda_{max}(a; \nu), \lambda_{max}(b; \nu)\} \mathbf{U_V} = c\mathbf{ID} \tag{6.2}$$
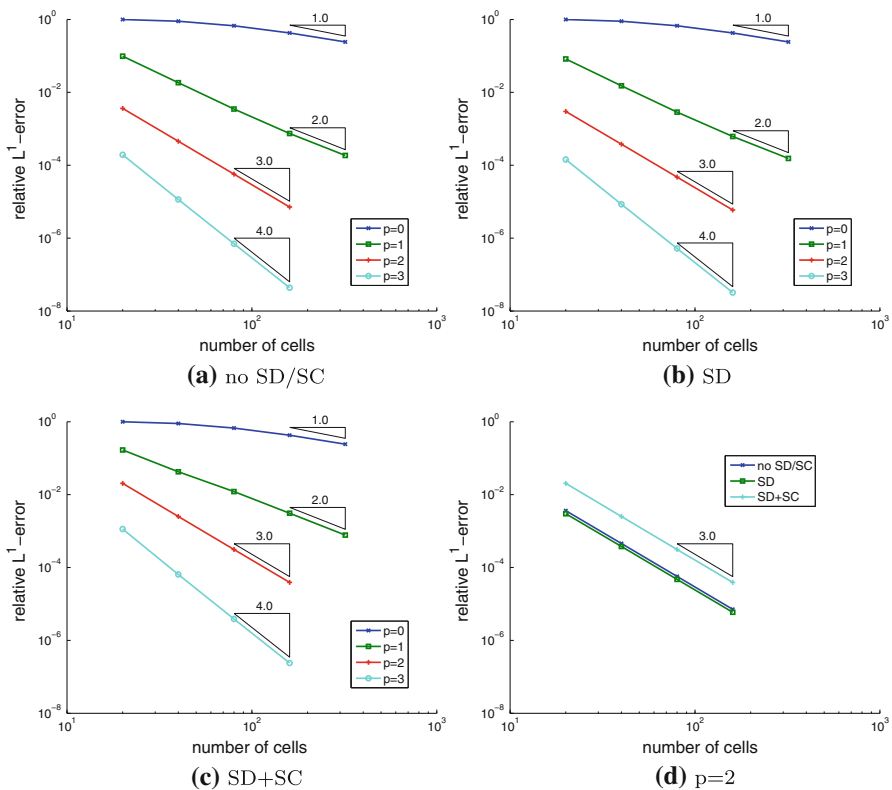
Fig. 1 Convergence for wave equation for smooth initial data

6.1.2 One space dimension

6.1.3 Smooth data: convergence rates.

In the first numerical experiment, we consider the wave equation in one space dimension and set the wave speed to be $c = 1$. The domain is $[-1, 1]$ with periodic boundary conditions and the initial data is

$$h = \sin(2\pi x), \quad u = \sin(2\pi x)/3.$$

As the initial data is smooth, the resulting solutions are smooth and we can compute the exact solution explicitly. The convergence rate for different schemes is provided in Fig. 1. As shown in the figure, we test the scheme (2.3) with different spaces of basis functions i.e, schemes with piecewise constant, linear, quadratic and cubic functions. Furthermore, different parts of the quasilinear form in (2.3) are tested. We test with just the DG form (2.4), with the DG and streamline diffusion form as well as with the full scheme (2.3). The results in Fig. 1 clearly show that the expected order of convergence is attained in this numerical experiment for all the choices of test functions as well
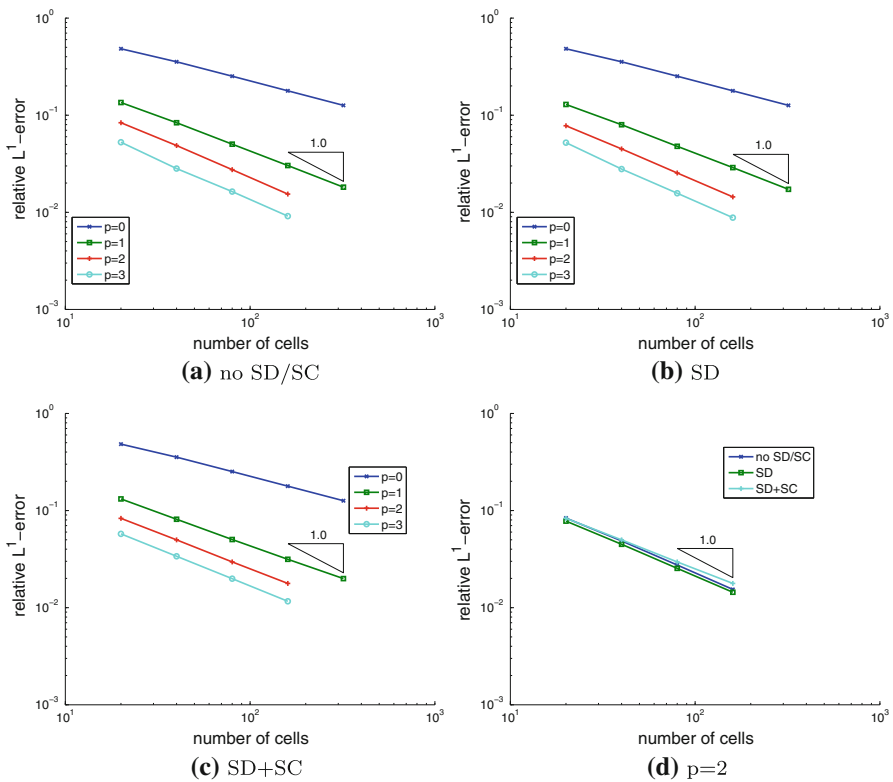
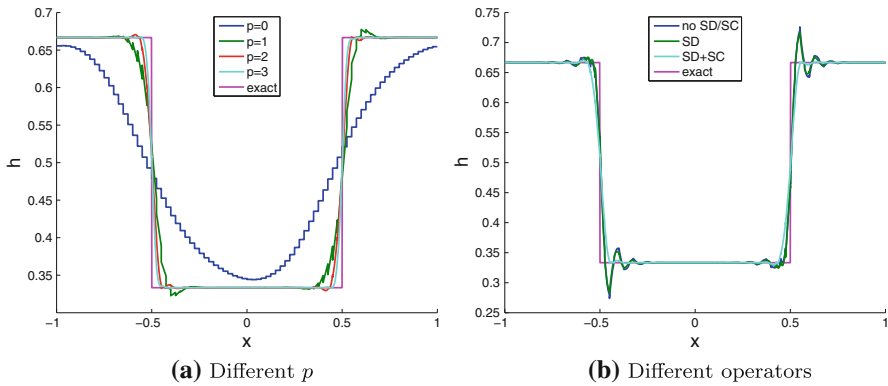**Fig. 2** Convergence for wave equation for discontinuous initial data

(a) Different p **(b)** Different operators

Fig. 3 Wave equation (h only), discontinuous initial data. *Left* SD + SC, different p, $N_c = 80$. *Right* Different operators, $p = 2$, $N_c = 80$

as for all constituent parts of the quasilinear form. The best results in this case are obtained with the pure DG form (2.4). Adding streamline diffusion (2.8) results in some smearing and adding the shock capturing form further smears the solution, even though the design order of accuracy is attained in all cases. The results indicate that the pure DG scheme will work quite well for smooth solutions.

6.1.4 Discontinuous data

In the next test, we consider the previous configuration with the one dimensional wave equation, but discontinuous initial data,

$$h(x, 0) = \begin{cases} 1.0, & \text{if } x < 0, \\ 0.0, & \text{if } x > 0. \end{cases} \quad u(x, 0) = \begin{cases} \frac{1}{3}, & \text{if } x < 0, \\ 0.0 & \text{if } x > 0. \end{cases}$$

In this case, the solutions are discontinuous. The rate of convergence for different schemes is shown in Fig. 2. As expected, the design order of accuracy is no longer achieved and the maximum rate of convergence is approximately 1. However, increasing the polynomial order does consistently result in lower errors for the same number of cells. Furthermore, time snapshots of the solution with different degrees of polynomial basis functions and with different operators are shown in Fig. 3. The figure clearly shows that there is considerable improvement in resolution when the polynomial degree is increased from piecewise constants to piecewise linears. There is a marginal improvement with further increase in the polynomial degree. Furthermore, the pure DG scheme (2.4) is clearly oscillatory near the discontinuities. The addition of streamline diffusion (2.8) damps the oscillations to some extent. However, the best results are obtained with the full scheme (2.3). The shock capturing term is clearly needed to reduce oscillations considerably. In fact, few visible oscillations remain when the shock capturing term is introduced.
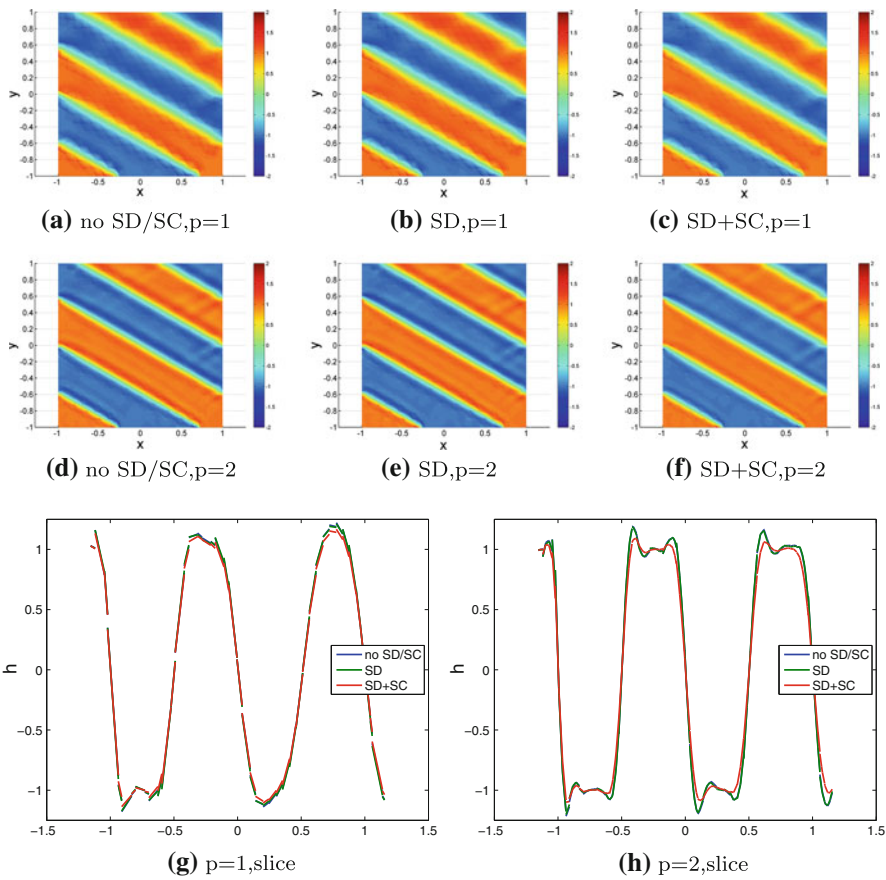
(a) no SD/SC,p=1   **(b)** SD,p=1   **(c)** SD+SC,p=1

(d) no SD/SC,p=2   **(e)** SD,p=2   **(f)** SD+SC,p=2

(g) p=1,slice   **(h)** p=2,slice

Fig. 4 2-D wave equation with discontinuous initial data and $N_c = 840$. The *bottom* panel shows cuts in α-direction through $(0, 0)$

6.1.5 Two space dimensions

Next we test the wave equations in a two dimensional domain $[-1, 1]^2$ with Dirichlet boundary conditions and with discontinuous initial data:

$$h = \text{sgn}(\sin(2\pi \langle \alpha, x \rangle)), \quad u = \alpha_1 \text{sgn}(\sin(2\pi \langle \alpha, x \rangle)), \quad v = \alpha_2 \text{sgn}(\sin(2\pi \langle \alpha, x \rangle)),$$
$$\text{where } \alpha = \begin{pmatrix} \cos(\pi/3) \\ \sin(\pi/3) \end{pmatrix}.$$

The numerical results with different schemes are provided in Fig. 4. The figure clearly shows that all the scheme resolve the solution quite well. There is clear gain in accuracy when piecewise quadratic basis functions are used instead of piecewise linears. As in the one dimensional tests, the pure DG and the DG-streamline diffusion schemes are oscillatory near discontinuities. However, these oscillations are significantly reduced with the addition of the shock capturing term (2.11a, 2.11b, 2.11c, 2.11d).

6.2 Euler equations

Next, we consider the nonlinear Euler equations of gas dynamics. In two space dimensions, they are of the form,

$$\mathbf{U}_t + \mathbf{F}^1(u)_x + \mathbf{F}^2(u)_y = 0,$$
$$\mathbf{U} = (\rho, \rho u, \rho v, \rho E),$$
$$\mathbf{F}^1(\mathbf{U}) = (\rho u, \rho u^2 + p, \rho u v, \rho u H)$$
$$\mathbf{F}^2(\mathbf{U}) = (\rho v, \rho u v, \rho v^2 + p, \rho v H) \tag{6.3}$$

Here, ρ is the density, u, v are the velocity fields and ρE is the total energy. Furthermore, auxillary quantities are the pressure p, sound speed c and the enthalpy H given by

$$p = (\gamma - 1)(\rho E - \frac{1}{2}\rho(u^2 + v^2)), \quad c = \sqrt{\gamma\frac{p}{\rho}}, \quad H = \frac{c^2}{\gamma - 1} + \frac{1}{2}(u^2 + v^2)$$

and γ is adiabatic exponent, which is set to 1.4 in all experiments.

Furthermore, the specific entropy $s = \log p - \gamma \log \rho$, the total entropy for the Euler equation is given by,

$$S = \frac{-\rho s}{\gamma - 1}$$

6.2.1 Entropy conservative fluxes, numerical diffusion operators

Here, we use the entropy conservative flux for the Euler equations derived in [18], see also [11].

The numerical diffusion operator is of the Rusanov type:

$$\mathbf{D}(a, b; v) = \max\{\lambda_{max}(a; v), \lambda_{max}(b; v)\}\, \mathbf{U_V}\left(\frac{a+b}{2}\right)$$
$$= \max\left\{c(a) + \left|u(a)v^1 + v(a)v^2\right|, c(b) + \left|u(b)v^1 + v(b)v^2\right|\right\} \mathbf{U_V}\left(\frac{a+b}{2}\right) \tag{6.4}$$

6.2.2 One space dimension

As a first numerical experiment, we consider the Euler equations in one space dimension on the domain $[-5, 5]$ and consider as initial data,

6.2.3 Sod shock tube

The Sod shock tube is a Riemann problem with left state $\rho = 1$, $u = 0$, $p = 1$ and right state $\rho = 0.125$, $u = 0$, $p = 0.1$. The results for the density and with piecewise quadratic basis functions are shown in Fig. 5. The results clearly show that the pure-DG and the DG and streamline diffusion schemes resolve the exact solution (consisting of a rarefaction, a contact and a shock) sharply but with oscillations near both the shock
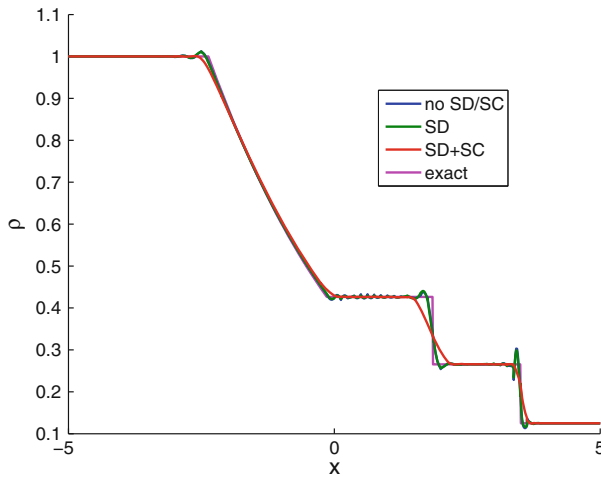
Fig. 5 Sod shock tube, density, different operators, $p = 2$, $N_c = 80$
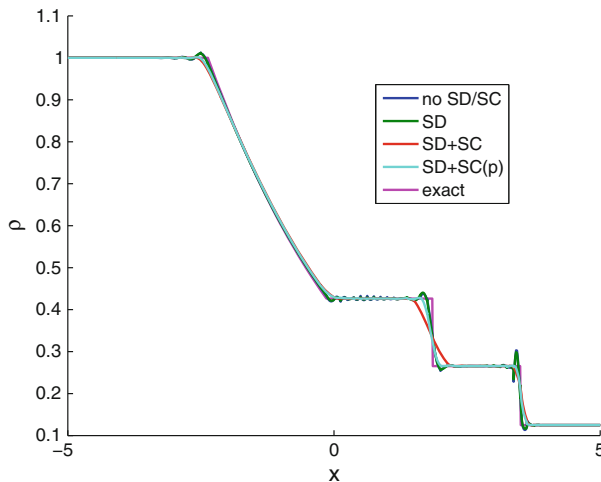

Fig. 6 Sod shock tube, density, different operators, $p = 2$, $N_c = 80$

wave as well as near the contact discontinuity. The addition of the shock capturing terms damps these oscillations considerably with some smearing at the shock wave. However, the contact discontinuity is smeared considerably and the results are quite unsatisfactory.

6.2.4 Pressure scaling

The above results indicate that we need some artificial compression near contacts. To this end, we utilize the fact that the pressure does not jump across contact
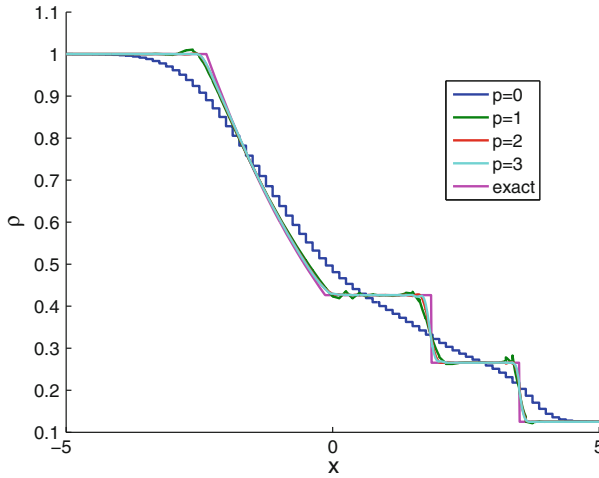
Fig. 7 Sod shock tube, density, SD+SC(p), different p, $N_c = 80$

discontinuities and modify the scaling (2.11b) in the shock capturing operator (2.11a, 2.11b, 2.11c, 2.11d) in the following way:

$$D_{n,K}^{SC} = \frac{D_{n,K}^p \left((\Delta x)^{1-\alpha} C^{SC} \overline{\text{Res}}_{n,K} + \Delta x^{\frac{1}{2}-\alpha} \bar{C}^{SC} \overline{\text{BRes}}_{n,K} \right)}{\sqrt{\int_{I^n} \int_K \left(\langle \mathbf{V}_t^{\Delta x}, \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}^{\Delta x}) \mathbf{V}_t^{\Delta x} \rangle + \sum_{k=1}^d \left\langle \mathbf{V}_{x_k}^{\Delta x}, \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}^{\Delta x}) \mathbf{V}_{x_k}^{\Delta x} \right\rangle \right) dx dt + \Delta x^\theta}},$$

(6.5)

$$D_{n,K}^p = \Delta x^2 \frac{\frac{1}{\Delta t^n} \frac{1}{|K|} \int_{I^n} \int_K \sqrt{\sum_{k=1}^d p_{x_k x_k}^2} dx dt}{\frac{1}{\Delta t^n} \frac{1}{|K|} \int_{I^n} \int_K p \, dx dt}$$

(6.6)

Hence, jumps in pressure serve as indicators of contact discontinuities and the shock capturing operator is switched off near the contacts. Note that introduction of the pressure scaling does not destroy the entropy stability of the scheme as the pressure term $D_{n,K}^p$ appears as a positive constant. Corresponding results for the Sod shock tube with the new pressure scaling and with piecewise quadratic basis functions are shown in Fig. 6. The figure clearly shows that the introduction of the new pressure scaling has significantly reduced the smearing at the contact while dampening oscillations near both the shock and the contact. Furthermore, the results for different polynomial degrees, shown in Fig. 7 demonstrate that there is a significant gain in accuracy when piecewise linears are used in place of piecewise constants. Gains in accuracy with even higher polynomial degrees are more modest.
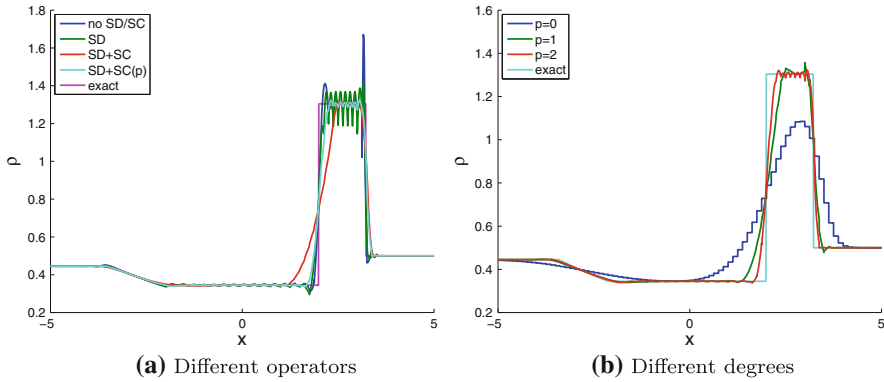
(a) Different operators   **(b)** Different degrees

Fig. 8 Lax shock tube, density, $N_c = 80$

6.2.5 Lax shock tube

Next, we consider the Lax shock tube, which is the Riemann problem with left state $\rho = 0.445, u = 0.698, p = 3.528$ and right state $\rho = 0.5, u = 0, p = 0.571$ on the domain $[-5, 5]$ with Dirichlet boundary conditions. The results are shown in Fig. 8. The figure shows that pure DG as well as the streamline diffusion-DG scheme are very oscillatory in the region between the shock and the contact discontinuity. The oscillations are decreased significantly when the shock capturing term is added but this term smears the contact discontinuity considerably. The sharpness at the contact discontinuity is regained once the pressure scaled shock capturing term is used, albeit with small oscillations in the intermediate state between the shock and the contact discontinuity. The behavior of the schemes with increasing polynomial degree in the basis functions is very similar to the Sod shock tube experiment.

6.3 Two space dimensions

6.3.1 Vortex advection

We consider the advection of an Euler vortex (see [11] for the setup) in the domain $[0, 10]^2$, with Dirichlet boundary conditions and a vortex centered at $x_c = 5, y_c = 5$ with $r_c = 1$ as initial condition:

$$u = 1 - (y - y_c)\phi(r), \quad v = 1 + (x - x_c)\phi(r), \quad \theta = 1 - \frac{\gamma - 1}{2\gamma}\phi(r)^2, \quad s = 0,$$

where $\theta = \frac{p}{\rho}, s = \log p - \gamma \log \rho, r = \sqrt{(x - x_c)^2 + (y - y_c)^2}, \phi(r)$ $= \epsilon e^{\left(\alpha\left(1 - \left(\frac{r}{r_c}\right)^2\right)\right)}, \epsilon = \frac{5}{2\pi}$ and $\alpha = 12$.

The exact solution is explicitly known in this case ([11] and references therein) and is a vortex propagating along the diagonal. We compute convergence rates
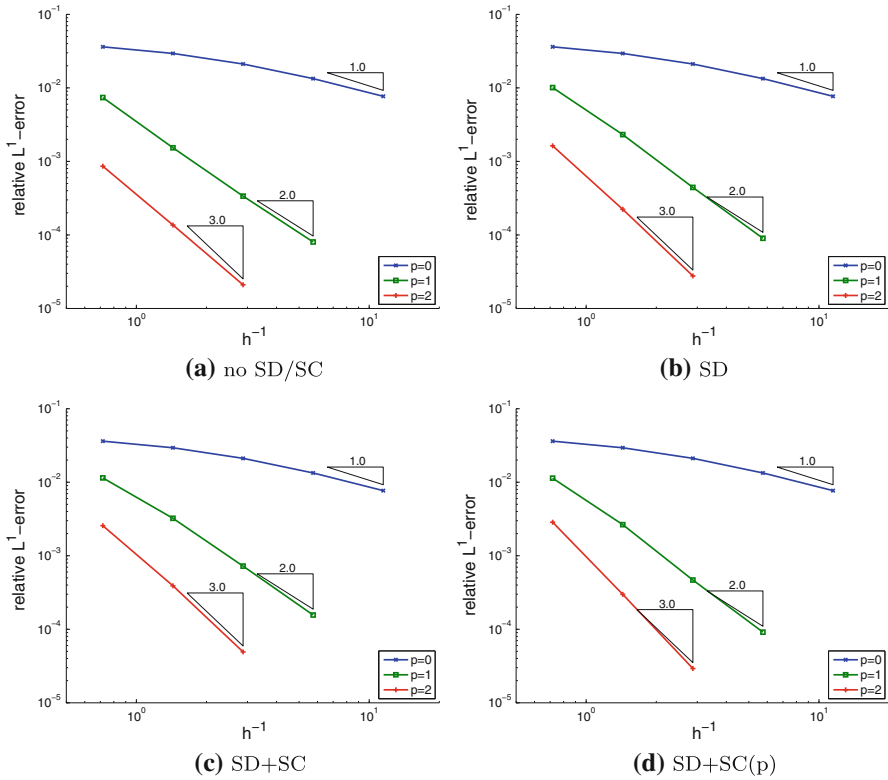
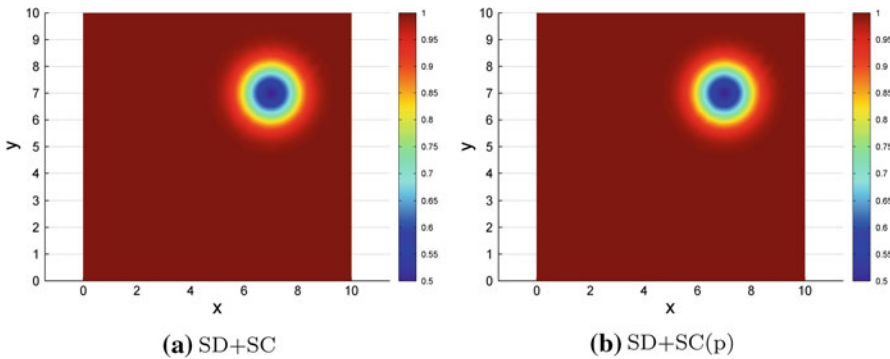**Fig. 9** Vortex-Advection, different p, different operators

Fig. 10 Vortex-Advection for the Euler equations, density $p = 2$, $N_c = 828$, $t = 2$

with different schemes and present the results in Fig. 9. The expected convergence rates for the piecewise constant, linear and quadratic basis functions are obtained. Furthermore, a time snapshot, shown in Fig. 10, demonstrates that the full scheme (with the pressure scaling) resolves the propagating vortex without excessive smearing.
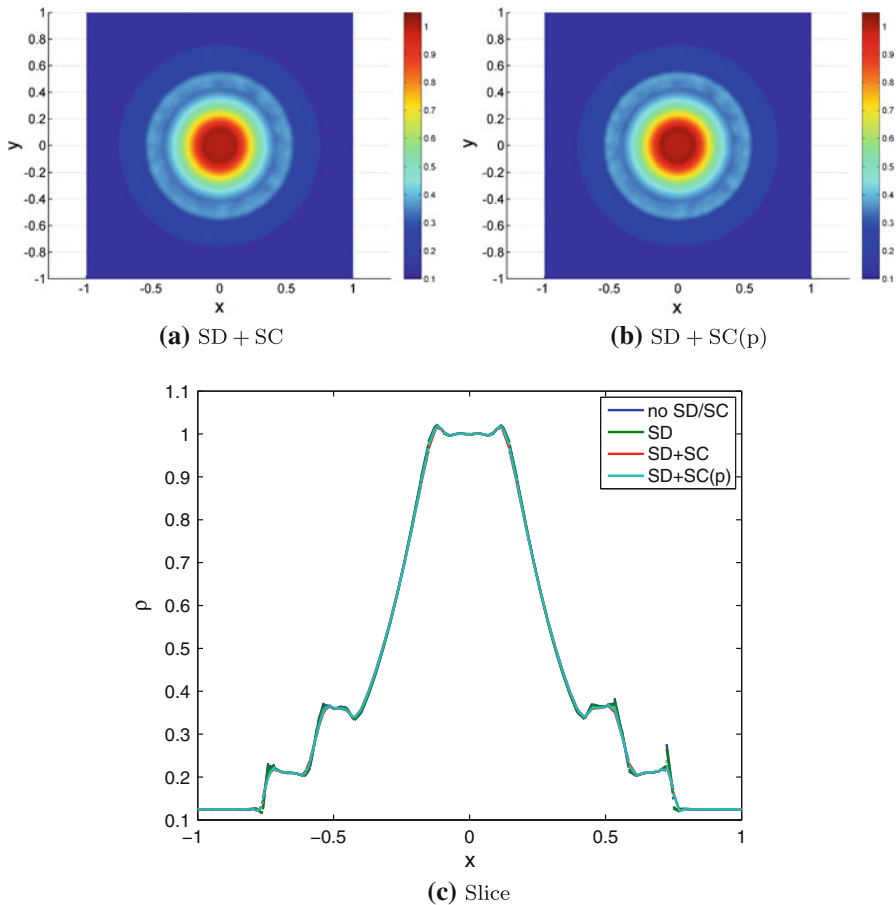
(a) SD + SC

(b) SD + SC(p)

(c) Slice

Fig. 11 Radial shock tube for Euler equations, density, $p = 1$, $N_c = 13{,}440$, $t = 0.2$

6.3.2 Radial shock tube

As a final numerical experiment, we consider the Euler equations in the domain $[-1, 1]^2$ with Dirichlet boundary conditions. The initial data consists of radial discontinuity at $r = \sqrt{x^2 + y^2} = 0.4$:

Outside this circle, the states are $\rho = 0.125$, $p = 0.125$ and inside the circle, the states $\rho = 1$, $p = 1$. Furthermore, $u = 0$, $v = 0$, everywhere in the domain initially.

The results with the scheme (2.3) are shown in Fig. 11 and demonstrate that the shock capturing (with pressure scaling) method works very well and the radial discontinuities are resolved sharply. One dimensional slices clearly show that adding shock capturing damps the oscillations generated with a pure DG or DG-streamline diffusion scheme.

7 Conclusion

In this paper, we consider systems of conservation laws (1.1) in several space dimensions and propose a streamline diffusion shock capturing spacetime DG method (2.3) to approximate them. The method is based on the following ingredients,

  (i) The entropy variables (rather than the conservative variables) serve as the degrees of freedom.
 (ii) The (spatial) numerical flux function is identical to the entropy stable fluxes (entropy conservative fluxes + numerical diffusion operators) proposed recently in the context of high-resolution finite volume schemes, [11] and references therein. In fact, these DG schemes can be thought as (implicit in time) high-order generalizations of the entropy stable schemes introduced by Tadmor in [27,28].
(iii) The streamline diffusion (2.8) and shock capturing operators (2.11a, 2.11b, 2.11c, 2.11d) are residual based and are needed to stabilize oscillations near shocks. In particular, a novel pressure scaling is introduced in order to modulate numerical diffusion near contact discontinuities.

We are able to show that the spacetime DG method satisfies the following (rigorous) properties:

 1. The fully discrete version of the method (with formal arbitrary order of accuracy) is shown to be entropy stable for a general system of conservation laws.
 2. We show that the spacetime DG method converges to an entropy measure valued solution of the underlying system of conservation laws. This result holds under the assumption that the approximate solutions are uniformly bounded in L^∞.
 3. The approximate solutions converge to the entropy solutions for scalar conservation laws and for linear symmetrizable systems.

Given the difficulties associated with showing convergence to weak solutions for nonlinear systems, our results on convergence to measure valued solutions are a crucial indication of the consistency of the proposed schemes. To the best of our knowledge, these are the first results concerning convergence to measure valued solutions for general nonlinear systems of conservation laws. Furthermore, we present extensive numerical experiments that indicate the desired order of accuracy is attained for smooth solutions. The schemes are shown to compute discontinuities like shocks and contact discontinuities robustly.

It is essential to view the results presented in this paper in the context of previous results. Streamline diffusion shock capturing DG methods were proposed about twenty years back in a number of papers, see [1,17,19–21] and references therein. We would like to compare our results with two of the papers that we believe are the most representative of the above list in the present context. First, [19] also considered streamline diffusion, shock capturing DG methods and we have used some of their results here. However, the presentation and analysis in [19] was mostly restricted to the scalar case. Here, we focus on nonlinear systems and view the current paper as generalizations of the results of [19] to systems of conservation laws. The next representative papers that we compare with are [1,2]. In these papers, the author considers systems

of conservation laws and presents a streamline diffusion shock capturing spacetime DG method to approximate them. The schemes of [1] are very similar to our proposed scheme (2.3). However, there are important differences. Firstly, the numerical flux functions proposed by us are based on the entropy stable schemes of Tadmor [27] in contrast to the system E-schemes employed by Barth [2]. Our DG scheme is thus a natural extension of entropy stable finite volume schemes to unstructured meshes and to arbitrarily high order of accuracy. Secondly, we introduce a pressure scaling in the shock capturing operator that is necessary to resolve contact discontinuities sharply. Furthermore, we go beyond the entropy stability analysis presented in [2] to prove that the streamline diffusion shock-capturing DG method converges to a entropy measure valued solution of the underlying conservation law. Thus, our results can be viewed as providing more rigorous analysis for the schemes presented in [1,2].

The current paper also poses interesting open questions. Our numerical experiments show that although the scheme (2.3) contains several free parameters, the results were not very sensitive to the choice of most of these parameters. The only exception was the parameter α in the shock capturing term (2.11b). As mentioned before, we observed a loss of accuracy when non-zero values of α were considered. The parameter α is only required for theoretical purposes, in the analysis of the entropy consistency (Theorem 4.2). We will discuss the role of α in a forthcoming paper [16] and show convergence to entropy solutions (in the scalar case) even when we set $\alpha = 0$.

At the level of computations, the most important issue is the efficient solution of the nonlinear system (5.6) at every time step. In the current paper, we employ a damped Newton method with a direct solver for the corresponding linear systems. This method is very expensive on account of the cost of forming the Jacobian. We plan to use a Newton–Krylov method to speed up computations. However, an efficient Newton–Krylov solver requires efficient preconditioners for the Krylov sub-steps. The design of such preconditioners is considered in a forthcoming paper.

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